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ON THE AUTOMORPHIC THETA REPRESENTATION FOR SIMPLY LACED GROUPS

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ABSTRACT

We construct an automorphic realization of the minimal representation of a split, simply laced group G, over a number field. The realization is by a residue, at a certain point, of an Eisenstein series, induced from the Borel subgroup. This residue representation is square integrable and defines the automorphic theta representation. It has "very few" Fourier coefficients, which turn out to have some extra invariance properties.

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Introduction

In this paper, we are concerned with the construction of an automorphic θ-module, for a simple, split, simply laced group G, defined over a number field F. By an automorphic θ-module, we mean a global $G_{\mathbb{A}}$-module of the form $\pi = \otimes\pi_\nu$, where π admits a $G_{\mathbb{A}}$-equivariant embedding into the space of automorphic forms on $G_{\mathbb{A}}$, and there is at least one local component π_ν with smallest Gelfand–Kirillov dimension (i.e. its Gelfand–Kirillov dimension is equal to one half of the dimension of the coadjoint orbit of highest weight in the Lie algebra). At a finite place ν, the unique, class-one, minimal representation (i.e. with smallest Gelfand–Kirillov dimension) was constructed in [K], [KS] and in [S]. If ν is archimedean, such a representation π_ν is considered in [V]. In this paper, we construct an automorphic realization of $\theta = \otimes\pi_\nu$, where each local component π_ν is class-one and of smallest Gelfand–Kirillov dimension. As in [K], we expect that all local components, of an automorphic θ-module, are minimal. We prove two properties of the automorphic theta representation, which manifest its rigid nature. Let G be of type $E_i, i = 6, 7, 8$, or of type D_m. Let Q be the maximal parabolic subgroup, whose Levi part L has semisimple part of type E_{i-1} or D_{m-1} respectively. (Here, E_5 means just D_5.) Let U be the unipotent radical of Q. (Note that U is abelian except in case E_8, where it is a Heisenberg group.) We show that the Fourier expansion, of the elements of θ, along U, consists of the constant term and one more orbit of characters under L_F, namely the orbit of the character which corresponds to the highest weight vector in $\mathrm{Lie}U_F$. Denote this character by χ_0. In case E_8, we consider the Fourier expansion, along U/Z, of the constant terms, along Z, of elements of the automorphic θ-representation. (Z is the center of U.) We get similar results. This is the content of Theorem 5.2. One more aspect of the rigidity of the automorphic θ-representation is the fact that the χ_0-Fourier coefficient of an element of θ is not only $\mathrm{Stab}_{L^0}(\chi_0)_F$-invariant, but also $\mathrm{Stab}_{L^0}(\chi_0)_{\mathbb{A}}$-invariant. (Theorem 5.4). Here $L^0 = [L, L]$. Actually, the proofs of Theorems 5.2 and 5.4 are valid for any automorphic θ-module (i.e. of the form $\otimes\pi_v$, such that at least one component, at a finite place, is the minimal representation).

In this paper, we realize θ as a residue representation of an Eisenstein series, coming from the Borel subgroup, and we prove that these residues lie in $L^2(G_k\backslash G_{\mathbb{A}})$ (Section 3). Moreover, θ has an inductive nature. The constant term of θ along U is, when restricted to $L^0_{\mathbb{A}}$, the direct sum of the trivial representation and the θ-representation of $L^0_{\mathbb{A}}$, as realized (automorphically) above. This is done in Section 2 and Section 3. In Section 6, we consider case D_m in more detail. We prove that the automorphic theta representation appears with multiplicity one. We conclude from this the equality of the space of theta with the explicit lift of the trivial representation of $\mathrm{SL}_2(\mathbb{A})$ to $G_{\mathbb{A}}$ (G of type $D_{\mathbb{A}}$), through the theta series kernel coming from the dual pair $(\mathrm{SL}_2, \mathrm{SO}_{2m})$ inside Sp_{2m} (rank $2m$).

Our main goal in constructing an automorphic θ module for G is to obtain a "theta kernel" which serves to define a lifting of automorphic forms between members of dual pairs inside G, and thereby obtain interesting examples of automorphic representations. For example, in [GRS], we have carried out such a program and considered the automorphic theta representation of $\widetilde{G}_2$, the three-fold cover of G_2. (This representation was constructed by Savin in [S1].) We considered the dual pairs (SL_3, Z_3) and $(\mathrm{SL}_2, \mathrm{SL}_2)$ inside G_2. The restriction of θ to $\widetilde{\mathrm{SL}}_3(\mathbb{A})$ (three fold cover) decomposes into a direct sum of irreducible automorphic representations, which are equivalent, at almost all places, to the theta representation of $\widetilde{\mathrm{SL}}_3$ (see [P.PS]). The restriction to the dual pair $\widetilde{\mathrm{SL}_2}(\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A})$ (three-fold cover for the first component) produces a decomposition of the form $\bigoplus \pi \otimes \theta(\pi)$, with π running over the cuspidal representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, and $\theta(\pi)$ being the so-called cubic lifting of π. In a sequel to this paper we shall give a decomposition of θ (of $G_{\mathbb{A}}$, G simple, split, simply laced) restricted to the dual pairs (G_2, L) inside G, and determine when a cuspidal representation π of $G_2(\mathbb{A})$ lifts (via the theta kernel) to a *cuspidal* representation $\theta(\pi)$ of $L_{\mathbb{A}}$. Here the phenomenon of a "tower of liftings" takes place exactly as in the classical cases of the symplectic groups and the orthogonal groups (see [R]), namely $\theta(\pi)$ is cuspidal, if and only if π has a zero lift, via the theta kernel, to the previous steps in the following two towers: $E_8 \supset E_7 \supset E_6 \supset D_5 \supset D_4$ or $E_8 \supset E_7 \supset D_6 \supset D_5 \supset D_4$. In particular, it is possible to obtain a partition of the space of cusp forms on $G_2(\mathbb{A})$, which is determined by an appropriate lifting from G_2 to L, or from L to G_2. This is the framework in which the construction of an automorphic θ-module and the study of its properties are important for us.

1. Notations

(1.1). We start by setting some notations for the exceptional groups of type E_6, E_7 and E_8. We assume that the groups are simply connected (and still denote them by E_i; moreover, we will use a convenient abuse of terminology and speak of the group E_i, while we really mean a group of type E_i). We will consider isogeneous groups as well. This will be explicitly mentioned in the text. We shall label the eight simple roots of E_8, α_i, $1 \leq i \leq 8$ as in [GS].

$$
\begin{array}{ccccccccccccc}
\alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 & & \alpha_8 \\
0 & \text{——} & 0 & \text{——} & 0 & \text{——} & 0 & \text{——} & 0 & \text{——} & 0 & \text{——} & 0 \\
 & & & & | & & & & & & & & \\
 & & & & 0 & & & & & & & & \\
 & & & & \alpha_2 & & & & & & & &
\end{array}
$$

Given a positive root α, we shall write $(n_1 \cdots n_8)$ for $\alpha = \sum_{i=1}^{8} n_i \alpha_i$ with $n_i \geq 0$. For the list of all positive roots, see [GS]. Given a root $\alpha = \sum_{i=1}^{8} n_i \alpha_i$ (positive or negative), x_α or $x_\alpha(r)$ or $x_{(n_1 \cdots n_8)}(r)$ will denote the one-dimensional unipotent subgroup corresponding to the root α. Since E_8 is a simply laced group, we have for all roots α and β

$$[x_\alpha(r_1), x_\beta(r_2)] = \begin{cases} x_{\alpha+\beta}(N_{1,2} r_1 r_2) & \alpha + \beta \text{ is a root,} \\ 1 & \text{otherwise.} \end{cases}$$

Here $N_{1,2} \in \{\pm 1\}$ and is chosen as in [GS] and $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$ where $g_1, g_2 \in E_8$.

We shall denote by w_{α_i} or w_i, $1 \leq i \leq 8$ the simple reflections in the Weyl group $W(E_8)$ of E_8, corresponding to the simple roots α_i. In short, we shall write $w(i_1 \cdots i_m)$ for $w_{i_1} w_{i_2} \cdots w_{i_m}$.

To each simple root, there is an embedding of SL_2 in E_8. Each such embedding gives a one-dimensional torus in E_8 corresponding to the torus $\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$ of SL_2. We shall denote the image of this torus in E_8, corresponding to the simple root α_i, $1 \leq i \leq 8$, by $h_i(t)$. Thus a general torus element is $\prod_{i=1}^{8} h_i(t_i)$, which we shall sometimes denote by $h(t_1, \ldots, t_8)$.

The action of the torus on the roots can be read from the Cartan matrix. Similarly, one can deduce the action of the Weyl group on the roots.

We shall consider the group E_7 embedded in the Levi part of the parabolic subgroup of E_8 obtained by deleting the root α_8. Similarly for E_6. It is embedded in the Levi part of the parabolic subgroup of E_8 (resp. E_7) obtained by deleting α_7 and α_8 (resp. α_7). We note that E_7 (resp. E_6) when regarded, as above, as a subgroup of E_8 (resp. E_8 or E_7) is still simply connected, so that our notation is consistent. See [BT] Cor. 4.4.

We shall also study the groups SO_{2m} which we define as follows. Let J_m denote the $m \times m$ matrix defined by $J_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. Define

$$\mathrm{SO}_{2m} = \{g \in \mathrm{GL}_{2m}\colon {}^t g J_{2m} g = J_{2m} \quad \text{and} \quad \det g = 1\}.$$

We will label the simple roots in D_m (simply connected) as follows:

$$\begin{array}{ccccccc} \beta_m & & \beta_4 & & \beta_3 & & \beta_1 \\ 0 & \text{------} \cdots & 0 & \text{------} & 0 & \text{------} & 0 \\ & & & & | & & \\ & & & & 0 & & \\ & & & & \beta_2 & & \end{array}$$

We will use similar notations as in the exceptional groups case. Thus if β is a positive root we will write $(n_1 \cdots n_m)$ for $\beta = \sum_{i=1}^m n_i\beta_i$. Also, if w_{β_i} or w_i are the simple reflections corresponding to the simple root β_i, we shall denote $w(i_1 \cdots i_r) = w_{i_1} \cdots w_{i_r}$. We will denote by $h_{\beta_i}(t)$, or simply $h_i(t)$, when there is no confusion, the one-dimensional torus corresponding to embedding of the SL_2 attached to the simple root β_i. We set

$$h(t_1, \ldots, t_m) = \prod_{i=1}^m h_i(t_i).$$

Let $O_{2m} = \{g \in \mathrm{GL}_{2m}\colon {}^t g J_{2m} g = J_{2m}\}$. D_m is a central double cover of $\mathrm{SO}'_{2m} = [O_{2m}, O_{2m}]$. Recall that over a field k, there is an exact sequence

$$1 \to \mathrm{SO}'_{2m}(k) \to \mathrm{SO}_{2m}(k) \to k^*/(k^*)^2 \to 1.$$

We will also need the action of the various Weyl groups on the torus. Since we are concerned with simply laced groups, we have, for simple roots α, β,

$$w_\alpha h_\beta(t) w_\alpha^{-1} = \begin{cases} h_\alpha(t^{-1}), & \beta = \alpha, \\ h_\beta(t) h_\alpha(t), & \langle \alpha, \beta\rangle = -1, \\ h_\beta(t), & \langle \alpha, \beta \rangle = 0. \end{cases}$$

In general, for a split reductive group G we denote by $\phi(G)$ the set of roots, by $\phi^+(G)$ the set of positive roots and by $\Delta(G)$ the set of simple roots. We will usually denote the highest root by β. Also, we sometimes denote by $B(G)$ (or just B) the Borel subgroup of G and, for a parabolic subgroup P, we denote by

ρ_P half the sum of positive roots which occur in the radical of P. For $P = B$, we let $\rho = \rho_B$. We use the standard analogous notation on the Lie algebra side. $\mathfrak{g} = \mathrm{Lie}(G)$; $\mathfrak{g}_\alpha$ denotes the root subspace corresponding to the root α, spanned by X_α.

Remark: The notation above and in the next subsection is meaningful when the group G is replaced by another simple group with the same Dynkin diagram.

(1.2). We will consider various maximal parabolic subgroups. More precisely, given a reductive algebraic group G we let $P(G) = M(G)V(G)$ denote the maximal parabolic subgroup of G whose unipotent radical is $V(G)$ and whose Levi part is given by

(a) $G = E_8$, $M(G) = \mathrm{GL}_1 \cdot E_7$,
(b) $G = E_7$, $M(G) = \mathrm{GL}_1 \cdot E_6$,
(c) $G = E_6$, $M(G) = \mathrm{GL}_1 \cdot D_5$, (almost direct products),
(d) $G = D_m$, $M(G) = \mathrm{GL}_1 \cdot A_{m-1}$.

We will need another parabolic subgroup of G for our constructions. Given G as above let $Q(G) = L(G)U(G)$ be the maximal parabolic subgroup of G whose Levi part is:

(a) $G = E_8$, $L(G) = \mathrm{GL}_1 \cdot E_7$,
(b) $G = E_7$, $L(G) = \mathrm{GL}_1 \cdot E_6$,
(c) $G = E_6$, $L(G) = \mathrm{GL}_1 \cdot D_5$,
(d) $G = D_m$, $L(G) = \mathrm{GL}_1 \cdot D_{m-1}$ (almost direct products).

Finally, consider the maximal parabolic subgroup $P_{\mathrm{Heis}}(G) = E(G)H(G)$, whose radical is a Heisenberg group. Its center is the root subgroup which corresponds to the highest root β.

(a) $G = E_8$, $E(G) = \mathrm{GL}_1 \cdot E_7$,
(b) $G = E_7$, $E(G) = \mathrm{GL}_1 \cdot D_6$,
(c) $G = E_6$, $E(G) = \mathrm{GL}_1 \cdot \mathrm{GL}_6$,
(d) $G = D_m$, $E(G) = \mathrm{GL}_1 \cdot (A_1 \times D_{m-2})$(almost direct products).

Remarks: (1) The group E_6 (resp. D_m for $m \geq 4$) has two associated parabolic subgroups with Levi part $\mathrm{GL}_1 \cdot D_5$ (resp. $\mathrm{GL}_1 \cdot A_{m-1}$). Since we will need both, we shall agree that $P(E_6)$ will denote the parabolic subgroup obtained by deleting the root α_6. If this is the case, Q will be the other maximal parabolic subgroup whose Levi part is $\mathrm{GL}_1 \cdot D_5$, the one obtained by deleting α_1. As for D_m, we shall agree that $P(D_m)$ will be obtained by deleting the root β_2 in whatever

D_m we choose. The other associated parabolic subgroup will be denoted by $P_a(D_m) = L_a(D_m)U_a(D_m)$. As for the other cases, $P(E_8) = Q(E_8)$ is obtained by deleting the simple root α_8. $P(E_7) = Q(E_7)$ is obtained by deleting α_7 and, finally, $Q(D_m)$ is obtained by deleting the root β_m.

(2) Notice that except in the case of E_8, $U(G)$ is an abelian group.

(3) We have $P_{\mathrm{Heis}}(G) = P(G)$ for $G = E_7, E_8$. $H(G)$ contains α_8 in case $G = E_8$, α_1 in case $G = E_7$, α_2 in case $G = E_6$ and β_{m-1} in case $G = D_m$.

For a maximal parabolic subgroup P of G we denote by α_P the unique simple root, which belongs to the radical of P.

We will need to study the space of double cosets $P(G)\backslash G/Q(G)$ when G equals D_m, E_6, E_7 or E_8. We denote the number of these double coset by $n(G)$.

LEMMA 1.1:

(a) *For $G = E_8$, $n(G) = 5$ and as representatives we may choose: e, $w(8)$, $w(876542345678)$, $w(87654231456734254316542345678)$, and the Weyl element w_0 with minimal length which sends all the roots in $U(E_8)$ to their negative.*

(b) *For $G = E_7$, $n(G) = 4$ and as representatives we may choose: e, w_7, $w(7654234567)$ and the Weyl element w_0 with minimal length which sends all roots in $U(E_7)$ to their negative.*

(c) *For $G = E_6$, $n(G) = 3$ and as representatives we may choose: e, $w(65431)$ and $w_0 = w(6543245613425431)$.*

(d) *For $G = D_m$, $n(G) = 2$ and as representatives we may choose: e, and $w_2\, w_3 \cdots w_m$.*

Proof: The proof is straightforward. It is clear that the representatives of $P(G)\backslash G/Q(G)$ can be chosen in $W(M(G))\backslash W(G)/W(L(G))$. We make a canonical choice. Namely we choose the representatives to be the Weyl elements w with minimal length mod $W(M(G))$ on the left and $W(L(G))$ on the right. In other words if $w = w_{i_1}w_{i_2}\cdots w_{i_\ell}$ with $\ell = \ell(w)$, then $w_{i_1} \notin W(M(G))$, $w_{i_\ell} \notin W(L(G))$ and using the relations among the simple reflections in $W(G)$, if $w = w_{j_1}w_{j_2}\cdots w_{j_\ell}$, then $i_1 = j_1$ and $i_\ell = j_\ell$. To find all such Weyl elements, we start by writing the long Weyl element w_0 in $W(G)/W(L(G))$, say $w_0 = w_{i_1}\cdots w_{i_\ell}$. Doing so, all the representatives of $W(M(G))\backslash W(G)/W(L(G))$ are all Weyl elements of the form $w_{i_j}\cdots w_{i_\ell}$ with $j \geq 1$ such that this word is minimal mod $W(M(G))$ on the left. (It is already minimal mod $W(L(G))$ on the right.) Let us illustrate these ideas in E_6. Recall that the only relations in $W(E_6)$ are the following. If α_i and α_j are not adjacent in the Dynkin diagram, then $w_{\alpha_i}w_{\alpha_j} = w_{\alpha_j}w_{\alpha_i}$, and if they are, then $w_{\alpha_i}w_{\alpha_j}w_{\alpha_i} = w_{\alpha_j}w_{\alpha_i}w_{\alpha_j}$. Also $w_{\alpha_i}^2 = 1$, for all α_i, $1 \le i \le 6$. Using these relations, we see that the long Weyl element in $W(E_6)/W(L(E_6))$ is $w_0 = w(6543245613425431)$. Indeed this Weyl element has length 16, which is the dimension of $U(E_6)$, and one can check that $w_0\alpha < 0$ if and only if α is a root in $U(E_6)$. Thus the minimal Weyl elements mod $W(M(E_6))$ on the left will be e, $w(65431)$ and w_0 itself. Hence these are the representatives of $W(M(E_6))\backslash W(E_6)/W(L(E_6))$. The other cases are done in a similar way. ∎

Remark: This lemma remains valid when G is replaced with a simple group of the same type (i.e. with the same Dynkin diagram). This is clear.

2. On poles of Eisenstein series

In this section, we will study the poles of certain Eisentein series on the groups $G = D_m$ for $m \ge 4, E_6, E_7$ and E_8. We will study the pole at a specific point. The method we use is as in [KR1], i.e. we will study the constant term along the unipotent radical $U(G)$ where G is as above. We will use the results of this section later on in the definition of the automorphic theta representation.

Let F be a number field and let $\mathbb{A}$ be its ring of adeles. Let $R = R(G)$ denote a maximal parabolic of G. Let δ_R denote the modular function of R. For $s \in \mathbb{C}$ set $I(s) = I(G,s) = \mathrm{Ind}_{R(\mathbb{A})}^{G(\mathbb{A})} \delta_R^{s+1/2}$. Consider the corresponding Eisenstein series defined first for $\mathrm{Re}(s)$ large, by

$$E_{R(G)}(g,f,s) = \sum_{\gamma \in R(F)\backslash G(F)} f(\gamma g, s)$$

for $g \in G(\mathbb{A})$ and $f \in I(s)$. This series converges absolutely for $\mathrm{Re}(s)$ large and admits a meromorphic continuation to the whole complex plane. It has a finite number of poles after a suitable normalization.

Let $K(G)$ be the standard maximal compact subgroup of $G(\mathbb{A})$. From now on, we shall restrict ourselves to standard sections f in $I(s)$. Thus, f is standard if it is $K(G)$ finite and its restriction to $K(G)$ is independent of s.

Given $f = \bigotimes_\nu f_\nu$ in $I(s)$, we denote by S the set of places such that f_ν is unramified for $\nu \notin S$. S may be the empty set. We denote by $\zeta_\nu(s)$ the local zeta factor at the place ν and we set $\zeta_S(s) = \prod_{\nu \notin S} \zeta_\nu(s)$.

Given a Weyl element $w \in W(G)$ we form the intertwining operator given, for $\mathrm{Re}(s) \gg 0$, by the intertwining integral,

$$(M_w(s)f)(g,s) = \int\limits_{N_w(\mathbb{A})} f(wng,s)dn$$

where N_w is the group generated by $\{x_\alpha(r)\colon \alpha > 0 \text{ and } x_{w\alpha}(r) \notin R\}$. Thus $M_w(s)$ is factorizable and $M_w(s) = \prod_\nu M_{w,\nu}(s)$. If f_ν is $K(G_\nu)$ fixed, normalized so that $f_\nu(e,s) = 1$, and $\widetilde{f}_\nu$ is the $K(G_\nu)$ fixed vector in the image of $M_{w,\nu}(s)$ normalized so that $\widetilde{f}_\nu(e,s) = 1$, then we have

$$M_{w,\nu}(s)f_\nu = L^1_\nu(w,s)\widetilde{f}_\nu.$$

Set

$$L^1_S(w,s) = \prod_{\nu \notin S} L^1_\nu(w,s).$$

We will also denote

$$A_w(s)f = \left(\prod_{\nu\in S} M_{w,\nu}(s)f_\nu\right) \otimes \prod_{\nu\notin S} \widetilde{f}_\nu.$$

Given G, R and f as above, we form the normalized Eisenstein series defined as follows. Denote by $L_S(G,R,s)$ the normalizing factor of $E_{R(G)}(g,f,s)$. We denote

$$E^*_{R(G)}(g,f,s) = L_S(G,R,s)E_{R(G)}(g,f,s).$$

By definition, $L_S(G,R,s)$ is the denominator of $L^1_S(w_0,s)$, when written as a quotient of products of zeta factors (after simplification), where w_0 is the representative of the big cell in $R\backslash G$ with minimal length.

To make things clearer, we start with two computational lemmas. The first lemma is easy to verify.

LEMMA 2.1: *We have:*

(1) $\delta_{P(E_8)}\Big(\prod_{j=1}^8 h_j(t_j)\Big) = |t_8|^{29}$,

(2) $\delta_{P(E_7)}\Big(\prod_{j=1}^7 h_j(t_j)\Big) = |t_7|^{18}$,

(3) $\delta_{P(E_6)}\Big(\prod_{j=1}^6 h_j(t_j)\Big) = |t_6|^{12}$,

(4) $\delta_{Q(E_6)}\Big(\prod_{j=1}^6 h_j(t_j)\Big) = |t_1|^{12}$,

(5) $\delta_{P(D_m)}\Big(\prod_{j=1}^m h_j(t_j)\Big) = |t_2|^{2m-2} \quad (m \geq 3),$

(6) $\delta_{Q(D_m)}\Big(\prod_{j=1}^m h_j(t_j)\Big) = |t_m|^{2m-2} \quad (m \geq 3).$

Next we compute the normalizing factors for certain Eisenstein series we use. We have:

LEMMA 2.2: *The factor $L_S(G,P,s)$ equals:*

(a) *If $G = D_m$*

$$\prod_{k=1}^{\left[\frac{m}{2}\right]} \zeta_S\Big((2m-2)s + m + 1 - 2k\Big).$$

(b) *If $G = E_6$*

$$\zeta_S(12s+6)\zeta_S(12s+3).$$

(c) *If $G = E_7$*

$$\zeta_S(18s+9)\zeta_S(18s+5)\zeta_S(18s+1).$$

(d) *If $G = E_8$*

$$\zeta_S(29s+29/2)\zeta_S(29s+19/2)\zeta_S(29s+11/2)\zeta_S(58s+1).$$

(e) *For $G = D_m$, the factor $L_S(G,Q,s)$ equals*

$$\zeta_S\Big((2m-2)s+1\Big)\zeta_S\Big((2m-2)s+m-1\Big).$$

Proof: The proof uses the method of Gindikin–Karpelevich as explained in [PSR1] Proposition 5.2. We rewrite their formula as follows. Let F_ν be a local field. Let R be a maximal parabolic subgroup in G. Parameterize the torus of G as $\prod_r h_r(t_r)$. Then there exist unique positive integers ℓ and k such that

$$\delta_R\Big(\prod_r h_r(t_r)\Big) = |t_\ell|^k, \quad t_r \in F_\nu^*.$$

The numbers ℓ and k can be read in our cases from Lemma 2.1. Let $w \in W(G)$, and $\alpha = \Sigma n_r\alpha_r$, if $G = E_6, E_7$ or E_8 and $\alpha = \Sigma n_r\beta_r$, if $G = D_m$, be a positive root. Then using Proposition 5.2 in [PSR1] we have:

$$\text{(2.1)} \qquad \int\limits_{N_w(F_\nu)} f_\nu(wn,s)dn = \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \frac{\zeta_\nu\Big(kn_\ell s + \frac{kn_\ell}{2} - \Sigma n_r\Big)}{\zeta_\nu\Big(kn_\ell s + \frac{kn_\ell}{2} + 1 - \Sigma n_r\Big)}.$$

Here f_ν is the unique K_ν fixed vector function in $\mathrm{Ind}_{R(F_\nu)}^{G(F_\nu)} \delta_R^{s+1/2}$ normalized so that $f_\nu(e,s) = 1$. To obtain Lemma 2.2 one can take $w = w_0$ to be the Weyl element in $W(R)\backslash W(G)$ as stated above. ∎

For $G = D_m, E_6, E_7, E_8$, let

$$s(G) = \begin{cases} \frac{m-3}{2m-2}, & G = D_m, \\ 1/4, & G = E_6, \\ 5/18, & G = E_7, \\ 19/58, & G = E_8. \end{cases}$$

There is a Weyl element w_0, such that

$$\chi_{sr}^{w_0} = \delta_B^{-1/2} \delta_P^{-s(G)+1/2}$$

where χ_{sr} is the real unramified character of the torus, which corresponds to the subregular unipotent orbit in ${}^L G$. See [S, p. 143]. For example, $w_0 = w(56), w(567), w(5678)$ in cases E_6, E_7, E_8 respectively.

We are now ready to prove:

THEOREM 2.3: *The Eisenstein series $E^*_{P(G)}(g,f,s)$ has at most a simple pole at $s = s(G)$ and it is obtained for some choice of section $f \in I(G,s)$.*

Proof: As was mentioned before, we will study the constant term of $E_{P(G)}(g,f,s)$ along U. The proof is similar to the cases studied in [KR1].

Let us briefly explain the idea. Write P, Q etc. for $P(G), Q(G)$ etc. For $g \in L(\mathbb{A})$ we have

$$\int_{U(F)\backslash U(\mathbb{A})} E_P(ug,f,s)du = \int_{U(F)\backslash U(\mathbb{A})} \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma ug, s)du.$$

For $h \in G(\mathbb{A})$ we have

$$\sum_{\gamma \in P(F)\backslash G(F)} f(\gamma h, s) = \sum_{w \in P(F)\backslash G(F)/Q(F)} \sum_{\delta \in (w^{-1}P(F)w \cap Q(F))\backslash Q(F)} f(w\delta h, s).$$

Since $Q = LU$ we can write $w^{-1}Pw \cap Q = L^w U^w$, where if X is a subgroup of Q then $X^w = w^{-1}Pw \cap X$. Thus L^w is a maximal parabolic subgroup of L. Write

$$\sum_{(w^{-1}P(F)w \cap Q(F))\backslash Q(F)} = \sum_{U^w(F)\backslash U(F)} \sum_{L^w(F)\backslash L(F)} .$$

Plugging all this in the constant term we obtain

$$\begin{aligned}
&\int\limits_{U(F)\backslash U(\mathbb{A})} E_P(ug,f,s)du \\
&=\sum_w \int\limits_{U(F)\backslash U(\mathbb{A})} \sum_{u^w\in U^w(F)\backslash U(F)} \sum_{\delta\in L^w(F)\backslash L(F)} f(wu^w u\delta g,s)du \\
&=\sum_w \int\limits_{U^w(F)\backslash U(\mathbb{A})} \sum_{\delta\in L^w(F)\backslash L(F)} f(wu\delta g,s)du.
\end{aligned}$$

Here w runs over $P(F)\backslash G(F)/Q(F)$. Factoring

$$\int_{U^w(F)\backslash U(\mathbb{A})} = \int_{U^w(\mathbb{A})\backslash U(\mathbb{A})} \int_{U^w(F)\backslash U^w(\mathbb{A})},$$

using the left invariance properties of f and setting the measure of $F\backslash\mathbb{A}$ to be one, we obtain

$$\begin{aligned}
&\int\limits_{U(F)\backslash U(\mathbb{A})} E_P(ug,f,s) = \sum_w \int\limits_{U^w(\mathbb{A})\backslash U(\mathbb{A})} \sum_\delta f(wu\delta g,s)du \\
&=\sum_w \sum_\delta (M_w(s)f)(\delta g,s) = \sum_w E_{L^w}(g,M_w(s)f,s').
\end{aligned} \tag{2.2}$$

Here w and δ are summed as above. Also, E_{L^w} is the Eisenstein series of the group L obtained by inducing from the maximal parabolic subgroup L^w and we understand that $E_{L^w}(g,M_w(s)f,s') = M_w(s)f$ if $L^w = L$. Finally, s' is some linear translation of s and we view $M_w(s)f$ as a section on L by restriction. We will now use this formula for the constant term for our cases. We start with:

(a): $G = D_m$ for $m \geq 4$. This case was actually done in [KR1]. We rewrite formula (1.2.14) in [KR1] as

$$\int\limits_{U(F)\backslash U(\mathbb{A})} E_{P(D_m)}(uh(a)g,f,s)du =$$

$$\begin{aligned}
&|a|^{(2m-2)(s+1/2)} E_{P(D_{m-1})}\Big(g,f,\frac{2m-2}{2m-4}s+\frac{1}{2m-4}\Big) \\
&+|a|^{-(2m-2)(s-1/2)} E_{P_a(D_{m-1})}\Big(g,M_v(s)f,\frac{2m-2}{2m-4}s-\frac{1}{2m-4}\Big).
\end{aligned} \tag{2.3}$$

Here $h(a) = h(a, a, a^2, \ldots, a^2)$ is a general element of the connected center of $L(D_m)$ and v is the second Weyl element as appears in Lemma 1.1(d). Also we view f and $M_v(s)f$ as sections on D_{m-1} by restriction. Notice that when $s = s(D_m)$ then

$$\frac{2m-2}{2m-4}s + \frac{1}{2m-4} = 1/2 \quad \text{and} \quad \frac{2m-2}{2m-4}s - \frac{1}{2m-4} = \frac{m-4}{2m-4} = s(D_{m-1}).$$

Thus by induction, $E^*_{P(D_{m-1})}\left(g, f, \frac{2m-2}{2m-4}s - \frac{1}{2m-4}\right)$ has a simple pole at $s = s(D_m)$ and the residue is the constant function. Following [KR1] we may deduce that after a suitable normalization, the second term on the right side of (2.3) can have at most a simple pole. Comparing the powers of $|a|$, we see that cancellations of the poles is not possible for $s = s(D_m)$, $m \geq 4$ and hence the theorem follows in this case.

(b): $G = E_6$. Once again, we compute the constant term along U to obtain (using Lemma 1.1)

$$\begin{aligned} \int\limits_{U(F)\backslash U(\mathbb{A})} E_{P(G)}(uh(a)g, f, s)du &= |a|^{24s+12} E_{Q(D_5)}\left(g, f, \frac{3}{2}s + \frac{1}{4}\right) \\ &+ |a|^{-12s+9} E_{P(D_5)}\left(g, M_v(s)f, \frac{12}{8}s - \frac{1}{8}\right) + |a|^{-48s+24}\left(M_{w_0}(s)f\right)(g, s). \end{aligned} \tag{2.4}$$

Here $h(a) = h(a^4, a^3, a^5, a^6, a^4, a^2)$, a "general" element of the center of the Levi part of $Q(E_6)$, and $g \in D_5$. Also, $v = w(65431)$ and w_0 is as defined in Lemma 1.1. Thus (2.4) is obtained by the general scheme as described in the beginning of the proof (see (2.2)). Let us sketch some of the details here. To obtain the first Eisenstein series we compute L^w for $w = e$. Thus $L^e = P \cap L = Q(D_5)$ and $U^e = P \cap U = U$. To compute s', we proceed as follows. First, by Lemma 2.1, we have $\delta_{Q(D_5)}\left(\prod_{j=2}^6 h_j(t_j)\right) = |t_6|^8$. Indeed, recall that now the D_5 is the subgroup of E_6 obtained by deleting α_1. On the other hand, $f\left(\prod_{j=2}^6 h_j(t_j)g, s\right) = |t_6|^{12(s+1/2)}$. Thus s' satisfies the equation $12(s+1/2) = 8(s'+1/2)$, i.e. $s' = \frac{3}{2}s + \frac{1}{4}$. The other cases are done in a similar way. Indeed when $v = w(65431)$ it is easy to check that $v^{-1}\alpha_1 = \alpha_3$; $v^{-1}\alpha_2 = (111100)$; $v^{-1}\alpha_3 = \alpha_4$; $v^{-1}\alpha_4 = \alpha_5$; $v^{-1}\alpha_5 = \alpha_6$ and $v^{-1}\alpha_6 < 0$. Thus the simple positive roots in the Levi part of L^v are α_1, α_3, α_4 and α_5 and hence $L^v = P(D_5)$. Since $v\alpha < 0$ for $\alpha = (100000)$; (101000); (101100); (101110)

and (101111) then $U^v\backslash U$ is the five-dimensional unipotent subgroup of U generated by these 5 roots. To compute s', we notice that by Lemma 2.2 we have $\delta_{P(D_5)}\left(\prod_{j=2}^{6} h_j(t_j)\right) = |t_2|^8$ and also

$$\int\limits_{U^v\backslash U} f\left(vu\prod_{j=2}^{6} h_j(t_j), s\right) du = |t_2|^{12(s+1/2)-3} \int\limits_{U^v\backslash U} f(vu, s)du$$

where $|t_2|^{-3}$ is obtained from the change of variables in $U^v\backslash U$. Thus $12(s+1/2) - 3 = 8(s'+1/2)$ which implies that $s' = \frac{12}{8}s - \frac{1}{8}$. Finally the computation of the powers of $|a|$ are done in a similar way. For example, in the case of $w = v = w(65431)$, one can check that $vh(a)v^{-1} = h(a, a^3, a^2, a^3, a, a^{-1})$ (see (1.1)). Also, we have a contribution of $|a|^{15}$ from the change of variables in $U^v\backslash U$. Since $\delta_{P(E_6)}(h(a, a^3, a^2, a^3, a, a^{-1})) = |a|^{-12}$ we obtain as the power of $|a|$ the number $-12(s+1/2) + 15 = -12s + 9$. It follows from (2.1) that for $\nu \notin S$

$$\left(M_{v,\nu}(s)f_\nu\right)(e, s) = \prod_{\substack{\alpha>0 \\ v^{-1}\alpha<0}} \frac{\zeta_\nu(12s + 6 - \Sigma n_r)}{\zeta_\nu(12s + 7 - \Sigma n_r)}.$$

Since the roots $\alpha > 0$ such that $v^{-1}\alpha < 0$ are 100000; 101000; 101100; 101110 and 101111 we see that

$$L_S^1(v, s) = \frac{\zeta_S(12s + 1)}{\zeta_S(12s + 6)}.$$

Taking into account the normalizing factors of the Eisenstein series appearing in Lemma 2.2 (see Lemma 2.2), we get (set $a = 1$)

(2.5)

$$\int\limits_{U(F)\backslash U(A)} E^*_{P(G)}(ug, f, s)du = E^*_{Q(D_5)}\left(g, f, \frac{3}{2}s + \frac{1}{4}\right)$$

$$+ E^*_{P(D_5)}\left(g, A_v(s)f, \frac{12}{8}s - \frac{1}{8}\right) + \zeta_S(12s - 2)\zeta_S(12s - 5) \cdot (A_{w_0}(s)f)(g, s).$$

Define

$$\left(A^*_{w_0}(s)f\right)(g, s) = \Big(\prod_{\nu\in S} \zeta_\nu(12s - 2)\zeta_\nu(12s - 5)\Big)^{-1} \left(A_{w_0}(s)f\right)(g, s).$$

Then

$$\zeta_S(12s - 2)\zeta_S(12s - 5)A_{w_0}(s) = \zeta(12s - 2)\zeta(12s - 5)A^*_{w_0}(s) \tag{2.6}$$

where $\zeta(s)$ denotes the complete zeta function i.e. $\zeta(s) = \prod_\nu \zeta_\nu(s)$. We need:

LEMMA 2.4: *Given $f \in I(s)$, the intertwining operators $A_v(s)$ and $A^*_{w_0}(s)$ are holomorphic at $s = s(G) = 1/4$.*

We will prove this lemma later.

Plugging (2.6) in (2.5) and computing the residue of (2.5) at $s = s(G) = 1/4$, we see that the factor $\zeta(12s-2)\ \zeta(12s-5)\ A^*_{w_0}(s)f$ has at most a simple pole at $s = 1/4$ and is nonzero for some choice of section f. Also $E^*_{P(D_5)}$ can have at most a simple pole at $s = 1/4$. This follows from case (a) when $G = D_5$. As in case (a) we may deduce that there is no cancellation of poles by comparing the power of $|a|$ at $s = 1/4$. Thus the theorem follows for this case.

(c): $G = E_7$. Here according to Lemma 1.1(b) there are 4 representatives for $P(G)\backslash G/Q(G)$. Let $v_1 = w_7$, $v_2 = w(7654234567)$ and w_0 as defined in Lemma 1.1(b). The constant term along U equals

$$\begin{aligned}
&\int\limits_{U(F)\backslash U(\mathbb{A})} E_{P(G)}(uh(a)g, f, s)du \\
&=|a|^{54s+27} f(g,s) + |a|^{18s+11} E_{P(E_6)}\Big(g, M_{v_1}(s)f, \frac{18}{12}s + \frac{2}{12}\Big) \\
&\quad + |a|^{-18s+11} E_{Q(E_6)}\Big(g, M_{v_2}(s)f, \frac{18}{12}s - \frac{2}{12}\Big) \\
&\quad + |a|^{-54s+27}\Big(M_{w_0}(s)f\Big)(g,s).
\end{aligned} \tag{2.7}$$

Here $h(a) = h(a^2, a^3, a^4, a^6, a^5, a^4, a^3)$ is in the center of $L(G)$ and $g \in E_6(\mathbb{A})$. Since $v_1^{-1}\alpha_i = \alpha_i$ for $1 \le i \le 5$, $v_1^{-1}\alpha_6 > 0$ and $v_1^{-1}\alpha_7 < 0$ we see that $L^{v_1} = P(E_6)$. Also it is clear that $U^{v_1}\backslash U$ is generated by the root $x_{\alpha_7}(r)$. As for v_2, we have $v_2^{-1}\alpha_1 > 0$; $v_2^{-1}\alpha_2 = \alpha_3$; $v_2^{-1}\alpha_3 = \alpha_2$; $v_2^{-1}\alpha_i = \alpha_i$ for $i = 4, 5, 6$ and $v_2^{-1}\alpha_7 < 0$. Thus $L^{v_2} = Q(E_6)$. Here $U^{v_2}\backslash U$ is the unipotent subgroup of U generated by the following 10 roots: (0000001); (0000011);(0000111); (0001111); (0011111); (0101111); (0111111); (0112111); (0112211) and (0112221). The points s' and the powers of $|a|$ are figured out as in case (b). Also as in case (b) one can easily check that

$$L^1_S(v_1, s) = \frac{\zeta_S(18s+8)}{\zeta_S(18s+9)}$$

and that

$$L^1_S(v_2, s) = \frac{\zeta_S(18s)\zeta_S(18s+4)}{\zeta_S(18s+5)\zeta_S(18s+9)}.$$

Multiplying (2.7) by $L_S(G,P,s)$ (see Lemma 2.2) and taking into account the normalizing factors of the Eisenstein series we obtain

$$\begin{aligned}
&\int\limits_{U(F)\backslash U(\mathbb{A})} E^*_{P(G)}(ug,f,s)du \\
(2.8)\qquad &= L_S(G,P,s)f(g,s) + \zeta_S(18s+1)E^*_{P(E_6)}\Big(g, A_{v_1}(s)f, \frac{18}{12}s+\frac{2}{12}\Big) \\
&+ \zeta_S(18s)E^*_{Q(E_6)}\Big(g, M_{v_2}(s)f, \frac{18}{12}s-\frac{2}{12}\Big) \\
&+ \zeta_S(18s)\zeta_S(18s-4)\zeta_S(18s-8)\Big(A_{w_0}(s)f\Big)(g,s).
\end{aligned}$$

Define

$$\Big(A^*_{w_0}(s)f\Big)(g,s) = \left(\prod_{\nu\in S}\zeta_\nu(18s)\zeta_\nu(18s-4)\zeta_\nu(18s-8)\right)^{-1}\left(A_{w_0}(s)f\right)(g,s).$$

Then

$$(2.9)\quad \zeta_S(18s)\zeta_S(18s-4)\zeta_S(18s-8)A_{w_0}(s) = \zeta(18s)\zeta(18s-4)\zeta(18s-8)A^*_{w_0}(s).$$

Later we will prove:

LEMMA 2.5: *Given $f\in I(s)$, the intertwining operators $A_{v_1}(s)$, $A_{v_2}(s)$ and $A^*_{w_0}(s)$ are holomorphic at $s=s(G)=5/18$.*

Next we plug (2.9) in (2.8) and compute the residue at $s=s(G)=5/18$. Notice that $\frac{18}{12}s+\frac{2}{12}>\frac{1}{2}$ for $s=5/18$ and that $\frac{18}{12}s(E_7)-\frac{2}{12}=s(E_6)$. As before, we get a nontrivial residue at $s(G)$ from the factor containing $\zeta(18s-4)$. Also from case (b) we see that $E^*_{Q(E_6)}$ can have at most a simple pole at $s=1/4$. Indeed, recall that in E_6 the parabolic subgroups P and Q are associated and hence the corresponding Eisenstein series share the same analytic properties. Once again, comparing the powers of $|a|$ we see that no cancellations are possible.

(d): $G=E_8$. In this case, we have 5 representatives for $P(G)\backslash G/Q(G)$. Let v_1,v_2 and v_3 denote the second third and fourth representatives as they appear in Lemma 1.1(a) and let w_0 be the element defined in Lemma 1.1. We have, for all $g \in E_7(\mathbb{A})$,

$$\begin{aligned} &\int\limits_{U(F)\backslash U(\mathbb{A})} E_{P(G)}(uh(a)g,f,s)du \\ =&|a|^{58s+29}f(g,s) + |a|^{29s+31/2}E_{P(E_7)}\Big(g, M_{v_1}(s)f, \frac{29}{18}s+\frac{1}{4}\Big) \\ &+|a|^{12}E_{P_{Heis}(E_7)}\Big(g, M_{v_2}(s)f, \frac{29}{17}s\Big) \\ &+|a|^{-29s+31/2}E_{P(E_7)}\Big(g, M_{v_3}(s)f, \frac{29}{18}s-\frac{1}{4}\Big) \\ &+|a|^{-58s+29}\Big(M_{w_0}(s)f\Big)(g,s). \end{aligned} \tag{2.10}$$

Here $h(a) = h(a^2, a^3, a^4, a^6, a^5, a^4, a^3, a^2)$ is in the center of $L(E_8)$. The case of v_1 is exactly as in the case of v_1 in $G = E_7$. For $w = v_2$, we have that $v_2^{-1}\alpha_1 > 0$; $v_2^{-1}\alpha_2 = \alpha_3$; $v_2^{-1}\alpha_3 = \alpha_2$; $v_2^{-1}\alpha_i = \alpha_i$, $i = 4,5,6,7$ and $v_2^{-1}\alpha_8 < 0$. Thus $L^{v_2} = Q'(E_7)$. Also $U^{v_2}\backslash U$ is the unipotent subgroup of U generated by the roots (00000001); (00000011); (00000111); (00001111) (00011111); (00111111); (01011111); (01111111); (01121111); (01122111); (01122211) and (01122221). As for v_3 we have: $v_3^{-1}\alpha_1 = \alpha_6$; $v_3^{-1}\alpha_2 = \alpha_2$; $v_3^{-1}\alpha_3 = \alpha_5$; $v_3^{-1}\alpha_4 = \alpha_4$; $v_3^{-1}\alpha_5 = \alpha_3$; $v_3^{-1}\alpha_6 = \alpha_1$ $v_3^{-1}\alpha_7 > 0$ and $v_3^{-1}\alpha_8 < 0$. Thus $L^{v_3} = P(E_7)$. Finally let us just mention that $U^{v_3}\backslash U$ is a maximal abelian subgroup of U. We omit the details.

We have:

$$L_S^1(v_1,s) = \frac{\zeta_S(29s+27/2)}{\zeta_S(29s+29/2)}$$

and

$$L_S^1(v_2,s) = \frac{\zeta_S(29s+7/2)\zeta_S(29s+1/2)}{\zeta_S(29s+29/2)\zeta_S(29s+19/2)}$$

and

$$L_S^1(v_3,s) = \frac{\zeta_S(29s-7/2)\zeta_S(29s+1/2)\zeta_S(29s+9/2)\zeta_S(58s)}{\zeta_S(29s+11/2)\zeta_S(29s+19/2)\zeta_S(29s+29/2)\zeta_S(58s+1)}.$$

We will also need the normalizing factor for the Heisenberg Eisenstein series in E_7:

$$L_S(E_7, P_{\mathrm{Heis}}, s) = \zeta_S(17s+17/2)\zeta_S(17s+11/2)\zeta_S(17s+7/2)\zeta_S(34s+1).$$

Multiplying (2.10) by $L_S(E_8, P, s)$ and taking into account the normalizing factor of the other terms we obtain

(2.11)
$$\int\limits_{U(F)\backslash U(\mathbb{A})} E^*_{P(G)}(ug,f,s)du$$
$$=L_S(G,P,s)f(g,s)+\zeta_S(58s+1)E^*_{P(E_7)}\Big(g,A_{v_1}(s)f,\frac{29}{18}s+\frac{1}{4}\Big)$$
$$+E^*_{Q'(E_7)}\Big(g,A_{v_2}(s)f,\frac{29}{17}s\Big)+\zeta_S(58s)E^*_{P(E_7)}\Big(g,A_{v_3}(s)f,\frac{29}{18}s-\frac{1}{4}\Big)$$
$$+\zeta_S(29s-9/2)\zeta_S(29s-17/2)\zeta_S(29s-27/2)\zeta_S(58s)\Big(A_{w_0}(s)f\Big)(g,s).$$

Denote the coefficient of $\Big(A_{w_0}(s)f\Big)(g,s)$ in (2.11) by $\overline{L}_S(w_0,s)$. Define

$$A^*_{w_0}(s)f=\Big(\prod_{\nu\in S}\overline{L}_\nu(w_0,s)^{-1}\Big)A_{w_0}(s)f.$$

Then $\overline{L}_S(w_0,s)A_{w_0}(s)=\overline{L}(w_0,s)A^*_{w_0}(s)$ where $\overline{L}(w_0,s)=\prod_\nu \overline{L}_\nu(w_0,s)$. Plugging this into (2.11) and arguing as in the previous cases we are done once we prove:

LEMMA 2.6: *Given $f\in I(s)$, the intertwining operators $A_{v_j}(s)$, $j=1,2,3$ and $A^*_{w_0}(s)$ are holomorphic at $s=s(G)=19/58$.*

To complete the proof of Theorem 2.3 we need:

Proof of Lemmas 2.4, 2.5 and 2.6: To prove these lemmas it is enough to show that given a Weyl element w and a place $\nu\in S$, the local intertwining operator $M_{w,\nu}(s)f_\nu$ is holomorphic at $s=s(G)$ for any choice of a local standard section $f_\nu\in \mathrm{Ind}^G_{P(G)}\delta^{s+1/2}_{P(G)}$. Here w is one of the Weyl elements appearing in those lemmas. First assume that w is v in case $G=E_6$ or v_1 or v_2 in case $G=E_7$ or w is v_1 or v_2 or v_3 in case $G=E_8$. Write $w=w(i_1)\cdots w(i_r)$ as a product of simple reflections such that $\ell(w)=r$. Thus

$$M_{w,\nu}=M_{w(i_r),\nu}\circ\cdots\circ M_{w(i_1),\nu}.$$

Due to this factorization it follows from the usual properties of GL_2-intertwining operators that the poles of $M_{w,\nu}$ are controlled by the poles of

$$\prod_{\substack{\alpha>0\\ w^{-1}\alpha<0}}\zeta_\nu\Big(kn_\ell s+kn_\ell/2-\Sigma n_r\Big). \tag{2.12}$$

Here $\alpha = \Sigma n_r \alpha_r$ and k and n_ℓ are given by Lemma 2.1. We will check on a case by case basis that for $s = s(G)$, $kn_\ell s + kn_\ell/2 - \Sigma n_r \geq 1$ for all $\alpha > 0$ with $w^{-1}\alpha < 0$. If $G = E_6$ then $w = v$ and $k = 12$ and $n_\ell = 1$ (since $v = w(65431)$ and, as mentioned in (b), the roots 100000; 101000; 101100; 101110 and 101111 are all roots $\alpha > 0$ with $v^{-1}\alpha < 0$). Thus, for $s = s(E_6) = 1/4$, $kn_\ell s + kn_\ell/2 - \Sigma n_r = 9 - \Sigma n_r \geq 1$, for all relevant roots. When $G = E_7$, we have for $s = s(E_7) = 5/18$ that $kn_\ell s + kn_\ell/2 - \Sigma n_r = 14n_7 - \Sigma n_r$. When $w = w_7$, $n_7 = 1$ and $\Sigma n_r = 1$ and $14n_7 - \Sigma n_r \geq 1$. For $w = v_2$, it is easy to check that $n_7 = 1$, and the highest root $\alpha > 0$, such that $v_2^{-1}\alpha < 0$, is the root 0112221, whose height is $\Sigma n_r = 9$. When $G = E_8$, we have $w = v_1, v_2, v_3$ and, for $s = s(E_8) = 19/58$, $kn_\ell s - kn_\ell/2 - \Sigma n_r \geq 24 - \Sigma n_r$. There are only five roots $\alpha > 0$, for which $24 - \Sigma n_r < 1$. They are (23454321); (23464321); (23465321); (23465421) and (23465431). However, one can check that for these roots $w^{-1}\alpha > 0$.

Next we study the intertwining operators $A_{w_0}(s)$. We start with $G = E_6$. For short write $A_{w_0}(s)f$ for $A_{w_0,\nu}(s)f_\nu$, where ν is a place in S and $f_\nu \in \mathrm{Ind}_{P(G)}^{G}\,\delta_{P(G)}^{s+1/2}$. Let $w = z(2)z(1)$, where $z(1) = w(431)$ and $z(2) = w_0 z(1)^{-1} = w(6543245613425)$. Thus

$$A_{w_0}(s)f = A_{z(1)}(s) \circ A_{z(2)}(s)f.$$

First, we claim that $A_{z(2)}(s)f$ is holomorphic at $s = s(G) = 1/4$. Indeed, as before, write $z(2) = w(6)w(5)\cdots w(2)w(5)$. Factoring $A_{z(2)}$ to GL_2-intertwining operators, we see that the poles of $A_{z(2)}(s)f$ are controlled by (2.12), with $w = z(2)$. Since $k = 12$ and $n_\ell = n_6 = 1$, (2.12) is given by

$$\prod_{\substack{\alpha>0 \\ z(2)^{-1}\alpha<0}} \zeta(9 - \Sigma n_r). \tag{2.13}$$

The highest roots $\alpha > 0$ with $z(2)^{-1}\alpha < 0$ are (111221) and (112211). For those, $\Sigma n_r = 8$. Thus (2.13) is holomorphic. By restriction, we have

$$A_{z(2)}\colon \mathrm{Ind}_{P(G)}^{G}\,\delta_{P(G)}^{s+1/2} \longrightarrow \mathrm{Ind}_{R(\mathrm{GL}_4)}^{\mathrm{GL}_4}\,\delta_R^{3s-1/2}.$$

Here $R(\mathrm{GL}_4)$ is the parabolic subgroup of GL_4 whose Levi part is $\mathrm{GL}_1 \times \mathrm{GL}_3$. Also GL_4 is embedded in G by deleting the roots α_2, α_5 and α_6. Finally, the simple positive roots in the Levi part of $R(\mathrm{GL}_3)$ are α_1 and α_3. Indeed, one can check that $z(2)\alpha_1 = \alpha_2$, $z(2)\alpha_3 = \alpha_4$ and $z(2)\alpha_4 > 0$. Thus, to prove our assertion, we need to show that

$$\zeta(12s-2)^{-1}\zeta(12s-5)^{-1}A_{z(1)}(s)f \tag{2.14}$$

is holomorphic at $s = s(G) = 1/4$ for all standard sections $f \in \mathrm{Ind}_{R(\mathrm{GL}_4)}^{\mathrm{GL}_4} \delta_R^{3s-1/2}$. Here, of course, we view $z(1) = w(431)$ as an element of GL_4 and similarly we view $A_{z(1)}$ as intertwining operator of GL(4) (see [PSR2]). To show this, we use Lemma 4.1 in [PSR2]. Let Σ denote the set of functions $f \in \mathrm{Ind}_{R(\mathrm{GL}_4)}^{\mathrm{GL}_4} \delta_R^{3s-1/2}$, such that the support of f is contained in $R(\mathrm{GL}_4)\overline{w}R(\mathrm{GL}_4)$, where

$$\overline{w} = \begin{pmatrix} &&&1\\ &&1&\\ &1&&\\ 1&&& \end{pmatrix}.$$

Lemma 4.1 in [PSR2] states that in order to study the poles of $A_{z(1)}(s)f(g,s)$ it is enough to consider $f \in \Sigma$ and also we may take $g = \overline{w}$. Since

$$z(1) = \begin{pmatrix} &1&&\\ &&1&\\ &&&1\\ 1&&& \end{pmatrix},$$

when viewed as a matrix in GL(4), we have

$$\begin{aligned} \left(A_{z(1)}(s)f\right)(\overline{w},s) &= \int_{F^3} f\left[z(1)\begin{pmatrix} 1&x&y&z\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}\overline{w}, s\right] dxdydz \\ &= \int_{F^3} f\left[\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ z&y&x&1 \end{pmatrix}, s\right] dxdydz. \end{aligned} \tag{2.15}$$

Write, for $z \neq 0$,

$$\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ z&&&1 \end{pmatrix} = \begin{pmatrix} 1&&&z^{-1}\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}\begin{pmatrix} z^{-1}&&&\\ &1&&\\ &&1&\\ &&&z \end{pmatrix}\overline{w}\begin{pmatrix} 1&&&z^{-1}\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}.$$

Plugging this in (2.15), conjugating and changing variables, we obtain

$$\left(A_{z(1)}(s)f\right)(\overline{w},s) = \int |z|^{12s-5} f\left(\overline{w}\begin{pmatrix} 1&x&y&z\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}, s\right) dxdyd^*z.$$

Note the multiplicative measure in z. Since $f \in \Sigma$ the poles of the last integral are controlled by the poles of $\int |z|^{12s-5}\varphi(z)d^*z$, where φ is a Schwartz–Bruhat function on F. The poles of the last integral are those of $\zeta(12s-5)$. Thus $\zeta(12s-5)^{-1}A_{z(1)}(s)f$ is holomorphic which clearly implies that (2.14) is holomorphic. This completes the case of $G = E_6$. The cases $G = E_7$ and $G = E_8$ are done similarly. When $G = E_7$ we write $w_0 = z(2)z(1)$ where $z(1) = w(4567)$ and $z(2) = w_0 z(1)^{-1}$. In this case $\ell(z(2)) = 23$. We have

$$A_{w_0}(s)f = A_{z(1)}(s) \circ A_{z(2)}(s)f.$$

As before, at $s = s(G) = 5/18$, $A_{z(2)}(s)$ is holomorphic. This is done by decomposing $A_{z(2)}(s)$ into GL(2)-intertwining operators and using (2.12). Also, we have

$$A_{z(2)}\colon \operatorname{Ind}_{P(G)}^{G} \delta_{P(G)}^{s+1/2} \longrightarrow \operatorname{Ind}_{R(\mathrm{GL}_5)}^{GL_5} \delta_R^{\frac{18}{5}s-\frac{13}{10}}.$$

Here $R(\mathrm{GL}_5)$ is the parabolic subgroup of GL(5) whose Levi part is $\mathrm{GL}_1 \times \mathrm{GL}_4$. Also GL_5 is embedded in E_7 by deleting the roots α_1, α_2 and α_3 and the simple positive roots in $R(\mathrm{GL}_5)$ are α_5, α_6 and α_7. As in the case of E_6 one can easily check that

$$\zeta(18s-8)^{-1}A_{z(1)}(s)f$$

is holomorphic which clearly implies the statement for this case. Finally, if $G = E_8$ we set $w_0 = z(2)z(1)$ where $z(1) = w(45678)$ and $z(2) = w_0 z(1)^{-1}$. Thus $\ell(z(2)) = 52$. We need to show that $\overline{L}_\nu(w_0, s)^{-1}A_{w_0,\nu}(s)$ is holomorphic at $s = s(G) = 19/58$ (see Lemma 2.6). We will show that

$$\zeta_\nu(29s - 27/2)^{-1}A_{w_0,\nu}(s)$$

is holomorphic at $s = 19/58$. As before we omit the reference to ν from the notations. Write $A_{w_0}(s) = A_{z(1)}(s) \circ A_{z(2)}(s)$. Factoring to GL_2-intertwining operators, we deduce that $A_{z(2)}(s)$ is holomorphic at $s = 19/58$. We also have

$$A_{z(2)}\colon \operatorname{Ind}_{P(G)}^{G} \delta_{P(G)}^{s+1/2} \longrightarrow \operatorname{Ind}_{R(\mathrm{GL}_6)}^{GL_6} \delta_R^{\frac{29}{6}s-\frac{17}{12}}.$$

Here $R(\mathrm{GL}_6)$ is the parabolic subgroup of GL(6) whose Levi part is $\mathrm{GL}_1 \times \mathrm{GL}_5$. Also GL_6 is embedded in E_8 by deleting the roots α_1, α_2 and α_3 and the simple positive roots in the Levi part of $R(\mathrm{GL}_6)$ are $\alpha_5, \alpha_6, \alpha_7$ and α_8. Finally, arguing as before, we obtain that

$$\zeta(29s - 27/2)^{-1}A_{z(1)}(s)$$

is holomorphic at $s = 19/58$. This completes the proof of Lemmas 2.4, 2.5 and 2.6. ∎

Remark 2.7: case of isogeneous groups. It is clear that the content of this section applies, with self-evident notation, to simple groups isogeneous to D_m, E_6, E_7, E_8. One constructs the analogous Eisenstein series, and Theorem 2.3 and the formulae in its proof remain valid, as the proof involves only the Dynkin diagrams. Moreover, we have the following simple situation. Denote, for this remark, by G^{sc} one of the simply connected groups above and by G a connected simple group of the same type; then we have an isogeny, defined over k, i: $G^{sc} \to G$. Let T denote the maximal torus of G. Then $G(k_\nu) = T(k_\nu)Im(i)(k_\nu)$, for all ν. Denote by P^{sc} and P two corresponding parabolic subgroups in G^{sc} and G respectively (i.e. they correspond to the same subset of simple roots). Clearly, for $p \in P^{sc}(k_\nu)$, $\delta_{P^{sc}}(p) = \delta_P(i(p))$. Consider the map i^*: $\mathrm{Ind}_{P_{\mathbb{A}}}^{G_{\mathbb{A}}} \delta^{s+1/2} \to \mathrm{Ind}_{P^{sc}_{\mathbb{A}}}^{G^{sc}_{\mathbb{A}}} \delta_{P^{sc}}^{s+1/2}$, induced by i. It is an isomorphism, which takes a right translation by g to a right translation by $i(g)$. Let φ_s be a section for $\mathrm{Ind}_{P_{\mathbb{A}}}^{G_{\mathbb{A}}} \delta_P^{s+\frac{1}{2}}$ and let $f_s = i^*(\varphi_s)$. For $\mathrm{Re}(s) \gg 0$, we have

$$\begin{aligned} E_{P^{sc}}(g, f_s) &= \sum_{\gamma \in P_k^{sc}\backslash G_k^{sc}} f_s(\gamma k) = \sum_{\gamma \in P_k^{sc}\backslash G_k^{sc}} \varphi_s(i(\gamma) i(g)) \\ &= \sum_{\gamma \in P_k \backslash G_k} \varphi_s(\gamma i(g)) = E_P(i(g), \varphi_s). \end{aligned} \tag{2.16}$$

Here we used Bruhat decomposition, which is "the same" for either G or G^{sc}, and hence $i\,(P_k^{sc}\backslash G_k^{sc})$ and $P_k\backslash G_k$ have the same set of representatives. ∎

3. The residue representation

Denote by θ'_G the space of automorphic forms on $G_{\mathbb{A}}$ obtained by $\mathrm{Res}_{s=s(G)} E^*_{P(G)}(g, f, s)$ as f varies in $I(s)$. θ'_G affords an automorphic representation of $G_{\mathbb{A}}$ by right translations. We denote this representation by θ'_G as well. From (2.3), (2.5), (2.8) and (2.11) we deduce

THEOREM 3.1: *We have*

$$\theta_G'^{U}\big|_{L^0(G)_{\mathbb{A}}} \subset \mathbb{1} \oplus \theta'_{L^0(G)}, \quad G \neq D_m,$$
$$\theta_{D_m}'^{U}\big|_{L^0(D_m)_{\mathbb{A}}} \subset \mathbb{1} \oplus \left(\theta'_{D_{m-1}}\right)^{\omega}, \quad G = D_m.$$

Here $L^0(G) = [L(G), L(G)]$ and $\theta_G'^U$ is the representation of $L^0(G)_{\mathbb{A}}$ by right translations on the space of constant terms along U of θ'_G, i.e. on

$$\left\{ g \mapsto \int_{U_F\backslash U_{\mathbb{A}}} \varphi(ug)du \colon g \in L^0(G)_{\mathbb{A}} \right\}.$$

$\left(\theta'_{D_{m-1}}\right)^{\omega}$ is the space of automorphic forms $\varphi(g^{\omega})$ on $D_{m-1}(\mathbb{A})$ where ω is the outer automorphism which flips β_1 and β_2. Note that in (2.3) the second term involves $P_a(D_{m-1})$. Since $P_a(D_{m-1}) = P(D_{m-1})^{\omega}$, it is clear that the second term in (2.3) lies in $(\theta'_{D_{m-1}})^{\omega}$. Note also that in (2.5), (2.6) and (2.11) $A^*_{w_0}(s)f|_{L^0(G)_{\mathbb{A}}}$ is constant at $s = s(G)$. The action of "general" central elements of $L(G)_{\mathbb{A}}$ on $\mathbb{1} \oplus \theta'_{L^0(G)}$ is read from (2.3), (2.4), (2.7), (2.10).

Proposition 3.2: *θ'_G consists of square integrable automorphic forms, i.e.*

$$\theta'_G \subset L^2(G(F)\backslash G(\mathbb{A})).$$

Proof: We use the square integrability criterion of [J]. See also [KRS, p. 520]. Since the elements $\phi \in \theta'_G$ are concentrated along the Borel subgroup $B(G)$, we have to show that the automorphic exponents of ϕ along $B(G)$ have real part which is a linear combination of the simple roots with negative coefficients. We check this case by case.

(1): $G = D_m, m \geq 4$. A successive application of (2.3) shows that the automorphic exponents along $B(G)$ correspond to the following characters of the adele points of the standard torus:

$$\chi_k \colon h(a_1, a_2, \ldots, a_m) \mapsto \delta_B^{-1/2}(h(a_1, \ldots, a_m))|a_k|^{k-2}|a_{k+1}|^{3-k}$$

for $4 \leq k \leq m$. (We define $a_{m+1} = 1$. Recall that for automorphic exponents it suffices to take $a_i \in \mathbb{A}^*$ with coordinate 1 at all finite places, and positive lying in the diagonal at archimedean places.) In additive form the character χ_k is expressed as

$$\mu_k = -\frac{1}{2} \sum_{\alpha \in \phi^+(G)} \alpha + \sum_{j=k+1}^{m} (m-j+1)\beta_j + (m-2) \left(\sum_{j=3}^{k} \beta_j + \frac{1}{2}(\beta_1 + \beta_2) \right).$$

The coefficient of β_i, $i > k$, in μ_k is $(m-i+1) - \frac{1}{2}(m+i-2)(m-i+1)$ which equals $-(m-i+1)(m+i) < 0$. If $3 \leq i \leq k$, the coefficient is $(m-2)$

$-\frac{1}{2}(m+i-2)(m-i+1) < 0$. If $i = 1, 2$, the coefficients is $\frac{m-2}{2} - \frac{1}{2}\frac{(m-1)m}{2} < 0$. Thus μ_k is a linear combination of all roots $\beta_1, \ldots, \beta_m$, with all coefficients negative.

(2): $G = E_6$. The automorphic exponents of θ'_G can be read off (2.4). The exponents which come from the second term of (2.4) correspond to the following character (using the previous case with $m = 5$):

$$\begin{aligned} h =& h\left(a^{4/3}, a, a^{5/3}, a^2, a^{4/3}, a^{2/3}\right) h(1, t_2, \ldots, t_6) \\ &\mapsto \delta_{Q(E_6)}^{-1/2}\left(h(a^{4/3}, a, a^{5/3}, a^2, a^{4/3}, a^{2/3}))|a|^2 \cdot \chi''(h(1, t_2, \ldots, t_6)\right) \end{aligned} \tag{3.2}$$

χ'' varies over the characters, which are trivial on $h\left(a^{4/3}, a, \ldots, a^{2/3}\right)$ and on $h(1, t_2, \ldots, t_6)$ correspond to the automorphic exponents of θ'_{D_5}, where D_5 is the semisimple part of $L(E_6)$, the Levi subgroup of $Q(E_6)$ (i.e. D_5 is based on the roots $\alpha_2, \ldots, \alpha_6$). Thus χ'' corresponds to a linear combination of $\alpha_2, \ldots, \alpha_6$ with negative coefficients. Recall that in (3.2) $a, t_2, \ldots, t_6$ are taken to have positive coordinates at the archimedean places and 1 at all other places. The element $h\left(a^{4/3}, a, \ldots, a^{2/3}\right)$ acts trivially by conjugation on $x_{\alpha_i}(r)$ for $2 \le i \le 6$ and takes $x_{\alpha_1}(r)$ to $x_{\alpha_1}(ar)$. Since $\delta_{Q(E_6)}(h(t_1, \ldots, t_6)) = |t_1|^{12}$, it is clear that the character (3.2) has the form $\chi'\chi''$, where, for h in (3.2), $\chi'(h) = |a|^{-6}$ and $\chi''(h) = \chi''(h(1, t_2, \ldots, t_6))$. We have $\chi'(h) = |a|^{-6} = \delta_{Q(E_6)}^{-3/8}(h)$. Thus χ' corresponds to a linear combination with negative coefficients of the roots which correspond to $U(E_6)$, the unipotent radical of $Q(E_6)$. Next, consider the automorphic exponents which come from the third term in (2.4). These have the form $\chi'\chi''$, where, for h in (3.2), $\chi'(h) = \delta_{Q(E_6)}^{-1/2}(h)|a|^4 = |a|^{-4} = \delta_{Q(E_6)}^{-1/4}$, and $\chi''(h) = \delta_{B(D_5)}^{-1/2}(h(1, t_2, \ldots, t_6))$, where $B(D_5)$ is the Borel subgroup of $D_5 \subset L(E_6)$. Thus $\chi'\chi''$ corresponds to a linear combination of the simple roots with negative coefficients.

(3): $G = E_7, E_8$. The proof here is as in the case of E_6. The exponents in each case are read off the last two terms of (2.7) and (2.10) respectively. In both cases, write an element of the torus (with positive coordinates at the archimedean places, and 1 at all finite places) as

$$h = h'h''$$

where

$$h' = \begin{cases} h(a, a^{3/2}, a^2, a^3, a^{5/2}, a^2, a^{3/2}), & G = E_7, \\ h(a^2, a^3, a^4, a^6, a^5, a^4, a^3, a^2), & G = E_8. \end{cases}$$

and

$$h'' = \begin{cases} h(t_1, \ldots, t_6, 1), & G = E_7, \\ h(t_1, \ldots, t_7, 1), & G = E_8. \end{cases}$$

Note that h' acts trivially (by conjugation) on $L(G)$, the Levi part of $Q(G)$, and it takes $x_{\alpha_7}(r)$ to $x_{\alpha_7}(ar)$, in case $G = E_7$, and $x_{\alpha_8}(r)$ to $x_{\alpha_8}(ar)$, in case $G = E_8$. The automorphic exponents of θ'_G (provided by (2.7), (2.10)) along B correspond to characters of the form

$$\chi = \chi'\chi''$$

where $\chi'(h'') \equiv 1$ and $\chi''(h') \equiv 1$. The last term in (2.7) (resp. in (2.10)) provides exponents which correspond to

$$\chi'(h') = \begin{cases} \delta_{Q(E_7)}^{-1/2}(h')|a|^6 = \delta_{Q(E_7)}^{-5/9}(h'), & G = E_7, \\ \delta_{Q(E_8)}^{-1/2}(h')|a|^{10} = \delta_{Q(E_8)}^{-\frac{19}{58}}(h'), & G = E_8. \end{cases} \tag{3.3}$$

(Note that $\delta_{Q(G)}(h'') \equiv 1$.)

$$\chi''(h'') = \delta_{B(E_{i-1})}^{-1/2}(h''), \qquad i = 7, 8 \tag{3.4}$$

where $B(E_{i-1})$ is the Borel subgroup of E_{i-1} realized as the semisimple part of $L(G) = L(E_i)$, the Levi part of $Q(G)$. It is clear, from (3.3), (3.4), that $\chi'\chi''$ corresponds to a linear combination of the simple roots with negative coefficients. (Every simple root has a negative coefficient.) The one before last term in (2.7) (resp. in (2.10)) provides exponents which correspond to

$$\chi'(h') = \begin{cases} \delta_{Q(E_7)}^{-1/2}(h')|a|^3 = \delta_{Q(E_7)}^{-7/18}(h'), & G = E_7 \\ \delta_{Q(E_7)}^{-1/2}(h')|a|^6 = \delta_{Q(E_8)}^{-23/58}, & G = E_8 \end{cases}$$

and $\chi''(h'')$ corresponds to an exponent of $\theta_{E_{i-1}}$, $i = 7, 8$ along $B(E_{i-1})$, so that it is a linear combination with negative coefficients of (all) simple roots $\alpha_1, \ldots, \alpha_{i-1}$ ($i = 7, 8$). χ' as a character of h corresponds to a linear combination with negative coefficients of the roots which occur in $U(G)$. ∎

Since θ'_G is square integrable, we get

COROLLARY 3.3: *θ'_G is a direct sum of irreducible (automorphic) representations.*

Remark 3.4: As in Remark 2.7, we have the same results for isogeneous groups. In the notation of Remark 2.7, we get the square integrable representations $\theta'_{G^{sc}}$ and θ'_G on $G^{sc}_{\mathbb{A}}$ and $G_{\mathbb{A}}$ respectively. These representations are the same in the sense that

$$\operatorname{Res}_{s=s(G)} E_{P^{sc}}(g, f_s) = \operatorname{Res}_{s=s(G)} E_P(i(g), \varphi_s).$$

Note also that $E_{P^{sc}}$ and hence $\theta'_{G^{sc}}$ have a trivial central character and hence $\theta'_{G^{sc}}$ is a representation of $i(G_{\mathbb{A}})$, and we have the following equality of automorphic representation of $i(G_{\mathbb{A}})$:

$$\theta'_{G^{sc}} = \theta'_G\big|_{i(G_{\mathbb{A}})}. \qquad \blacksquare$$

4. Definition of the (automorphic) theta representation

Let F be a local nonarchimedean field. Let G be one of the groups E_6, E_7, E_8. (We will treat D_m separately.) In [KS] the minimal representation $\theta_{G(F)} = \theta_G$ (simply laced, simply connected group in general) is defined. It is first constructed as an irreducible unitary representation of the parabolic subgroup $P_{\text{Heis}}(G) = E(G)\cdot H(G)$, and then it is proven to extend to a representation θ_G of G. Let ψ be a nontrivial character of F, and let σ_ψ be the Stone–von Neumann representation of $H(G)$, with central character ψ (we identify the center of $H(G)$ as $t \mapsto x_\beta(t) = \exp(tX_\beta)$, $t \in F$). σ_ψ extends to $P^0_{\text{Heis}}(G) = E^0(G)H(G)$, where $E^0(G)$ is the semisimple part of $E(G)$. Then we have

$$\widehat{\theta}_G\big|_{P^0_{\text{Heis}}} = \operatorname{Ind}_{P^0_{\text{Heis}}}^{P_{\text{Heis}}} \sigma_\psi. \tag{4.1}$$

$\widehat{\theta}_G$ denotes the unitary completion of the smooth representation θ_G. The r.h.s. of (4.1) is an induction in the L^2-sense. From [KS], it follows that

$$\theta_G \subset \operatorname{Ind}_{P_{\text{Heis}}}^G \delta_{P_{\text{Heis}}}^{-z(G)+\frac{1}{2}}$$

where

$$z(G) = \begin{cases} 7/22, & G = E_6, \\ 11/34, & G = E_7, \\ 19/58, & G = E_8. \end{cases} \tag{4.2}$$

Note that

$$\delta_{P_{\text{Heis}}}(h(t_1, \ldots, t_i)) = \begin{cases} |t_2|^{11}, & G = E_6 (i = 6), \\ |t_1|^{17}, & G = E_7 (i = 7), \\ |t_8|^{29}, & G = E_8 (i = 8). \end{cases} \tag{4.3}$$

The representation $\mathrm{Ind}_{P_{\mathrm{Heis}}}^{G} \delta_{P_{\mathrm{Heis}}}^{-z(G)+\frac{1}{2}}$ has a unique irreducible subrepresentation. This is shown in [S]. Thus θ_G is the unique irreducible subrepresentation of $\mathrm{Ind}_{P_{\mathrm{Heis}}}^{G} \delta_{P_{\mathrm{Heis}}}^{-z(G)+\frac{1}{2}}$, and by duality, since θ_G is self-dual,

PROPOSITION 4.1: *θ_G is the unique irreducible quotient of* $\mathrm{Ind}_{P_{\mathrm{Heis}}}^{G} \delta_{P_{\mathrm{Heis}}}^{z(G)+\frac{1}{2}}$.

Now let us show

PROPOSITION 4.2: *θ_G is the unramified subquotient of* $\mathrm{Ind}_{P(G)}^{G} \delta_{P}^{s(G)+\frac{1}{2}}$. *(For the definition of $s(G)$, see Section 2.)*

Proof: In case $G = E_8$, $P = P_{\mathrm{Heis}}$ and $s(E_8) = z(E_8)$, and so there is nothing to prove. Assume $G = E_6, E_7$.

Let

$$w = \begin{cases} w(2456), & G = E_6, \\ w(134567), & G = E_7. \end{cases} \tag{4.4}$$

Note that the positive roots α, such that $w(\alpha) < 0$, are roots in $V(G)$, the unipotent (abelian) radical of $P(G)$. More precisely,

(4.5)

$$\begin{aligned} \phi_w &= \{\alpha \in \phi^+ \mid w(\alpha) < 0\} \\ &= \begin{cases} \{\alpha_6, \alpha_6+\alpha_5, \alpha_6+\alpha_5+\alpha_4, \alpha_6+\alpha_5+\alpha_4+\alpha_2\}, & G = E_6, \\ \{\alpha_7, \alpha_7+\alpha_6, \alpha_7+\alpha_6+\alpha_5, \alpha_7+\alpha_6+\alpha_5+\alpha_4, & \\ \alpha_7+\alpha_6+\alpha_5+\alpha_4+\alpha_3, \alpha_7+\alpha_6+\alpha_5+\alpha_4+\alpha_3+\alpha_1\}, & G = E_7. \end{cases} \end{aligned}$$

Consider, first in the convergence domain, the intertwining operator $M_w(z)$ on $\mathrm{Ind}_{P_{\mathrm{Heis}}}^{G} \delta_{P_{\mathrm{Heis}}}^{z+\frac{1}{2}}$

$$M_w(z) f_z(g) = \int f(w \Pi_{\alpha\in\phi_w} x_\alpha(r_\alpha) g) \Pi_{\alpha\in\phi_w} dr_\alpha \tag{4.6}$$

for a holomorphic section f_z in $\mathrm{Ind}_{P_{\mathrm{Heis}}}^{G} \delta_{P_{\mathrm{Heis}}}^{z+\frac{1}{2}}$. Clearly

$$M_w(z) f_z(ug) = M_w(z) f_z(g), \quad u \in V(G). \tag{4.7}$$

Now restrict g to be in $M(G)$. Let $M^0(G)$ be the semisimple part of $M(G)$. If $G = E_6$, then $M^0(G) = D_5$, which is based on the simple roots $\{\alpha_1, \ldots, \alpha_5\}$. If $G = E_7$, then $M^0(G) = E_6$, which is based on the simple roots $\{\alpha_1, \ldots, \alpha_6\}$. Consider the parabolic subgroup $P_{\mathrm{Heis}}(M^0(G))$ of $M^0(G)$. It is easy to check that w takes the simple roots γ, which belong to the Levi part of $P_{\mathrm{Heis}}(M^0(G))$, to simple roots which belong to the Levi part of $P_{\mathrm{Heis}}(G)$, and so for such simple roots γ, we have

$$\delta_{P_{\mathrm{Heis}}(G)}(x_{\pm\gamma}(t)) \equiv 1.$$

Also, the radical of $P^0_{\mathrm{Heis}}(M^0(G))$ is taken by w to the radical of $P_{\mathrm{Heis}}(G)$. Thus

$$M_w(z)f_z\big|_{M^0(G)} \in \mathrm{Ind}_{P^0_{\mathrm{Heis}}(M^0(G))}^{M^0(G)} \delta_{P^0_{\mathrm{Heis}}(M^0(G))}^{z'+\frac{1}{2}}.$$

To compute z', we check the effect of left translation in g in (4.6) by $h_3(t)$ in case $G = E_6$ and by $h_2(t)$ in case $G = E_7$. We have, in case $G = E_6$,

$$\begin{aligned} M_w(z)f_z(h_3(t)g) &= |t|^{-2}\int f(wh_3(t)\Pi_{\alpha\in\phi_w}x_\alpha(r_\alpha)g)\Pi dr_\alpha \\ &= |t|^{-2}\int f(h_2(t)h_4(t)h_3(t)w\Pi_{\alpha\in\phi_w}x_\alpha(r_\alpha)g)\Pi dr_\alpha \\ &= |t|^{-2}\delta_{P_{\mathrm{Heis}}(E_6)}^{\frac{1}{2}+z}(h_2(t))M_w(z)f_z(g) \\ &= |t|^{\frac{7}{2}+11z} = \delta_{P_{\mathrm{Heis}}(M^0(G))}^{\frac{1}{2}+\frac{11}{7}z}(h_3(t)). \end{aligned}$$

Thus

$$z' = \frac{11}{7}z. \tag{4.8}$$

Similarly, in case $G = E_7$,

$$z' = \frac{17}{11}z. \tag{4.9}$$

Now, when we formally substitute $z = z(G)$ in (4.8) and (4.9) we get

$$z' = 1/2$$

and hence

$$M_w(z(G))f\big|_{M^0(G)} \in \mathrm{Ind}_{P_{\mathrm{Heis}}(M^0(G))}^{M^0(G)} \delta_{P_{\mathrm{Heis}}(M^0(G))}^{1/2+1/2} \tag{4.10}$$

for f in $\mathrm{Ind}_{P_{\mathrm{Heis}}(G)}^{G} \delta_{P_{\mathrm{Heis}}}^{z(G)}$. To justify this step, we show that $M_w(z)f_z$ is holomorphic and not identically zero for $z = z(G)$. We have the factorization

$$M_w(z) = \begin{cases} M_{w_6}(z_6)M_{w_5}(z_5)M_{w_4}(z_4)M_{w_2}(z_2), & G = E_6 \\ M_{w_7}(z_7)M_{w_6}(z_6)\cdots M_{w_3}(z_3)M_{w_1}(z_1), & G = E_7 \end{cases}$$

for appropriate linear functions z_i of z. We view this factorization for operators defined on $\mathrm{Ind}_{B(G)}^{G}\,\delta_{P_{\mathrm{Heis}}(G)}^{\frac{1}{2}+z}$. Examining the analytic properties of each factor $M_{w_i}(z_i)$ is a "GL_2-calculation". Indeed, we just have to consider

$$\int_F \widetilde{f}(w_\alpha x_\alpha(r))dr,$$

for a simple root α, and $\widetilde{f}$ in an appropriate induced representation from $B(G)$. This is the Gindikin–Karplevich method. Thus, the poles of $M_w(z)$ are contained in those of

$$\prod_{\substack{\alpha>0\\ w^{-1}(\alpha)<0}} \zeta(\langle -\rho+(z+\frac{1}{2})2\rho_{P_{Heis}(G)},\alpha\rangle) = \begin{cases} \prod_{j=1}^{4}\zeta(11z+j+\frac{1}{2}), & G=E_6,\\ \prod_{j=2}^{7}\zeta(17z+j+\frac{1}{2}), & G=E_7.\end{cases}$$

Clearly, $z=\frac{7}{22}$ in case $G=E_6$ and $z=\frac{11}{34}$ in case $G=E_7$ are points of holomorphicity. Moreover, when $M_w(z)$ is applied to the normalized K-fixed vector in $\mathrm{Ind}_{P_{\mathrm{Heis}}(G)}^{G}\,\delta_{P_{\mathrm{Heis}}}^{z+1/2}$, evaluated at 1, we get

$$\prod_{\substack{\alpha>0\\ w^{-1}(\alpha)<0}} \frac{\zeta(\langle -\rho+(z+\frac{1}{2})2\rho_{P_{Heis}(G)},\alpha\rangle)}{\zeta(\langle -\rho+(z+\frac{1}{2})2\rho_{P_{Heis}(G)},\alpha\rangle+1)},$$

which is nonzero for $z=z(G)$. This justifies (4.10). Since

$$\mathrm{Ind}_{P_{\mathrm{Heis}}(M^0(G))}^{M^0(G)}\,\delta_{P_{\mathrm{Heis}(M^0(G))}}^{1/2+1/2}$$

has $1_{M^0(G)}$ as a quotient, then composing $M_w(z(G))$ with a map

$$T'\colon \mathrm{Ind}_{P_{\mathrm{Heis}}(M^0(G))}^{M^0(G)}\,\delta_{P_{\mathrm{Heis}}(M^0(G))}^{1/2+1/2} \to 1_{M^0(G)},$$

and using (4.7), we obtain an $M^0(G)$-map

$$T\colon J_{V(G)}\left(\mathrm{Ind}_{P_{Heis}}^{G}\,\delta_{P_{Heis}}^{z(G)+1/2}\right) \longrightarrow 1_{M^0(G)}. \tag{4.11}$$

$J_{V(G)}$ denotes the Jacquet functor. In order to see how T transforms the action of $M(G)$, it remains to check this on the following central elements of $M(G)$ (see proof of Theorem 2.3):

$$h(a) = \begin{cases} h(a^2,a^3,a^4,a^6,a^5,a^4), & G=E_6,\\ h(a^2,a^3,a^4,a^6,a^5,a^4,a^3), & G=E_7.\end{cases}$$

Note that $h(a)$ commutes with (the roots in) $M^0(G)$ and it acts on $x_{\alpha_6}(r)$ by $x_{\alpha_6}(a^3r)$ in case $G = E_6$. It acts on $x_{\alpha_7}(r)$ by $x_{\alpha_7}(a^{-1}r)$ in case $G = E_7$. It is easy to check that the action of $h(a)$ through T is (in both cases) by $|a|^{12} = \delta_{P(G)}^{\frac{1}{2}-s(G)}(h(a))$. This and (4.11) imply by Frobenius reciprocity that there is a nontrivial G-equivariant map

$$\tau\colon \operatorname{Ind}_{P_{\mathrm{Heis}}}^{G} \delta_{P_{\mathrm{Heis}}}^{z(G)+1/2} \to \operatorname{Ind}_{P(G)}^{G} \delta_{P(G)}^{-s(G)+1/2}.$$

Now $\operatorname{Ind}_{P_{\mathrm{Heis}}}^{G} \delta_{P_{\mathrm{Heis}}}^{z(G)+1/2}$ is generated by f^0 — the unramified element as a G-module (since it has a unique quotient which is unramified, i.e. θ_G) — and hence the image of τ is generated as a G-module by $\tau(f^0)$. $G\cdot\tau(f^0)$ has, of course, a unique irreducible quotient which is unramified, and since the quotient is also one for

$$\operatorname{Ind}_{P_{\mathrm{Heis}}(G)}^{G} \delta_{P_{\mathrm{Heis}}(G)}^{z(G)+1/2},$$

it must be θ_G. Thus θ_G is the unramified constituent of $\operatorname{Ind}_{P(G)}^{G} \delta_{P(G)}^{-s(G)+1/2}$, and similarly, by duality, θ_G is the unramified constituent of $\operatorname{Ind}_{P(G)}^{G} \delta_{P(G)}^{s(G)+1/2}$. ∎

Remark 1: It is certain that θ_G is the unique quotient of $\operatorname{Ind}_{P(G)}^{G} \delta_{P(G)}^{s(G)+1/2}$ (i.e. $G\cdot\tau(f^0)$ is irreducible), but for our needs Proposition 4.2 will suffice.

Remark 2: Let G denote a group of type E_6, E_7, E_8, and again, denote by G^{sc} the corresponding simply connected group. (In case E_8, $G = G^{sc}$, and in cases E_6, E_7, G can be either simply connected or of adjoint type.) Denote $s_0 = s(G^{sc})$. Consider as in Remark 2.7 the representations $\tau = \operatorname{Ind}_P^G \delta^{s_0+\frac{1}{2}}$ and $\tau^{sc} = \operatorname{Ind}_{P^{sc}}^{G^{sc}} \delta^{s_0+\frac{1}{2}}$ (now over F) and the natural isomorphism $i^*\colon \tau \to \tau^{sc}$. Let $f_0 \in \tau$ be the unramified vector, and let $V = \tau(G)\cdot f_0$ (the G-module generated by f_0). V has a unique unramified quotient $W\backslash V$. i^* induces a vector space isomorphism $W\backslash V \simeq i^*(W)\backslash i^*(V)$. We have $i^*(V) = i^*(\tau(G)f_0) = \tau^{sc}(G^{sc})i^*(f_0)$. This is due to the fact that f_0 and $i^*(f_0)$ are the unramified vectors of τ and τ^{sc} respectively, and that $G = i(B^{sc})K$, K being the maximal compact subgroup of G. Similarly, $i^*(W)$ is G^{sc}-invariant. Let us show that $W\backslash V$ is irreducible over $i(G^{sc})$. Indeed $i(G^{sc})$ is normal in G and $i(G^{sc})\backslash G$ is finite and abelian. Decompose over $i(G^{sc})$, $W\backslash V = \bigoplus_\omega \overline{\tau}(\omega)(W\backslash V_0) = \bigoplus_\omega W\backslash\tau(\omega)V_0$, where ω varies over a subset of the set of representatives of $i(G^{sc})\backslash G$, and $W\backslash V_0$ is an irreducible subspace of $W\backslash V$ over $i(G^{sc})$. This induces a decomposition (as G^{sc}-modules) $i^*(W)\backslash i^*(V) = \bigoplus_\omega i^*(W)\backslash i^*(\tau(\omega)V_0)$. Since $i^*(W)\backslash i^*(V)$ has a unique unramified quotient, $f_0 + W$ must project onto one summand only in $\bigoplus_\omega (W\backslash\tau(\omega)V_0)$, say it is $W\backslash V_0$. But now

$$W\backslash V = \overline{\tau}(G)\cdot(f_0+W) = W\backslash\tau(G)\cdot f_0 = W\backslash\tau(i(G^{sc}))f_0 = W\backslash V_0.$$

Let us denote by $\theta_{G(F)}$ the unramified subquotient of τ. We will abbreviate and denote θ_G. We have shown that θ_G is irreducible over $i(G^{sc})$ and that $\mathrm{Res}_{i(G^{sc})}\theta_G = \theta_{G^{sc}}$. In particular, it follows that θ_G is a minimal representation of G, since the character distribution of either θ_G or $\theta_{G^{sc}}$ on $\mathrm{Lie}(G) = \mathrm{Lie}(G^{sc})$ is exactly the same. See [S, section 2].

Let us consider the case D_m. Here, let us use the more familiar notation Spin to denote the simply connected group. Over the local field F, we have the exact sequences

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
1 \arrow[r] & Z_2 \arrow[r] & \mathrm{Spin}(F) \arrow[r, "j"] & \mathrm{SO}'_{2m}(F) = [O_{2m}, O_{2m}] \arrow[r] \arrow[d, "\nu"] & 1 \\
 & & & \mathrm{SO}_{2m}(F) \arrow[d] & \\
 & & & F^*/(F^*)^2 &
\end{tikzcd}$$

Consider the Howe lift (local theta correspondence) of the trivial representation of $\mathrm{SL}_2(F)$ to $\mathrm{SO}_{2m}(F)$, i.e. the lift via the Weil representation for the dual pair $\mathrm{SL}_2 \times O_{2m}$ inside Sp_{2m} (rank $2m$). The result of the lift does not depend on an additive character of F. Exactly as in [KR, Section 3], this representation is irreducible, unramified and embeds into $\mathrm{Ind}_{P(\mathrm{SO}_{2m})}^{\mathrm{SO}_{2m}(F)}\delta_P^{-s_0+\frac{1}{2}}$. It is a unique irreducible subrepresentation ($s_0 = s(D_m)$). It can be realized as the space of functions on $\mathrm{SO}_{2m}(F)$

$$h \mapsto \omega_\psi(1,h)\phi(0,0) \tag{4.12}$$

where ϕ is a Schwartz function on $X\oplus X$, and X is a maximal isotropic subspace of the $2m$-dimensional quadratic space on which $\mathrm{SO}_{2m}(F)$ acts and preserves the quadratic space on which $\mathrm{SO}_{2m}(F)$ acts and preserves the quadratic form;

$\omega_\psi(g,h)$ is the Weil representation of $\widetilde{\mathrm{Sp}}_{2m}(F)$ restricted to $\mathrm{SL}_2(F)\times \mathrm{SO}_{2m}(F)$, and ψ is a nontrivial character of F. Let $\theta_{\mathrm{SO}_{2m}}$ be the unramified quotient of $\mathrm{Ind}_{P(\mathrm{SO}_{2m})}^{\mathrm{SO}_{2m}(F)}\delta_P^{s_0+\frac12}$. It is also realized as the space of functions (4.12). By the above diagram $\theta_{\mathrm{SO}_{2m}}$ defines unramified representations of $\mathrm{Spin}(F)$ and $\mathrm{SO}'_{2m}(F)$. Exactly as in Remark 2, these representations are irreducible (one over $\mathrm{Spin}(F)$ and one over $\mathrm{SO}'_{2m}(F)$) and form the unique unramified quotients θ_{Spin} and $\theta_{\mathrm{SO}'_{2m}}$ of $\mathrm{Ind}_{P(\mathrm{Spin})}^{\mathrm{Spin}(F)}\delta_P^{s_0+\frac12}$ and $\mathrm{Ind}_{P(\mathrm{SO}'_{2m})}^{\mathrm{SO}'_{2m}(F)}\delta_P^{s_0+\frac12}$ respectively. We relaxed the notation a little bit. We also have $\mathrm{Res}_{v\circ j(\mathrm{Spin}(F))}\theta_{\mathrm{SO}_{2m}}=\theta_{\mathrm{Spin}}$ and $\mathrm{Res}_{v(\mathrm{SO}'_{2m}(F)}\theta_{\mathrm{SO}_{2m}}=\theta_{\mathrm{SO}'_{2m}}$. By [S, Theorem 2.2] $\theta_{\mathrm{Spin}},\theta_{\mathrm{SO}'_{2m}}$ and $\theta_{\mathrm{SO}_{2m}}$ are minimal representations of $\mathrm{Spin}(F)$, $\mathrm{SO}'_{2m}(F)$ and $\mathrm{SO}_{2m}(F)$, respectively. It follows from the realization (4.12) that

$$\theta^{\omega}_{\mathrm{SO}_{2m}}\cong\theta_{\mathrm{SO}_{2m}},\qquad \text{for } \omega=\begin{pmatrix} I_{m-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{m-1}\end{pmatrix}. \tag{4.13}$$

(Indeed $\omega_\psi(1,h)$ in (4.12) is meaningful for $h\in O_{2m}(F)$.) Conjugation by ω induces on Spin, the outer automorphism which flips the roots β_1 and β_2. Still denoting it by ω, we get that $\theta^{\omega}_{\mathrm{Spin}_{2m}}\cong\theta_{\mathrm{Spin}_{2m}}$.

We are ready to construct an automorphic realization for θ_G. Let F be a number field and G of type D_m,E_6,E_7,E_8 (simply connected or not). Consider, for each place v of F, the cyclic G_v-module θ'_ν generated by the unramified vector f^0_ν in $\mathrm{Ind}_{P(G_\nu)}^{G_\nu}\delta_{P(G_\nu)}^{s(G)+\frac12}$. We have a map from $\tau_G=\otimes\theta'_\nu$ to square integrable automorphic forms on $G_{\mathbb{A}}$, defined by the residue at $s=s(G)$ of the Eisenstein series, which corresponds to $\mathrm{Ind}_{P(G_{\mathbb{A}})}^{G_{\mathbb{A}}}\delta_{P(G_{\mathbb{A}})}$ (see Section 3). Thus we consider only sections which are generated by the $K_{\mathbb{A}}$-fixed vector $f^0=\otimes f^0_\nu$. Denote by θ_G the space of automorphic forms obtained for such sections by the residues at $s=s(G)$.

THEOREM 4.3: *θ_G is irreducible and, at all finite places ν, the local component of θ_G is θ_{G_ν}.*

Proof: θ_G is an invariant subspace of θ'_G and hence, by Corollary 3.2 and Remark 3.4, $\theta_G=\bigoplus\pi^{(i)}$, a direct sum of irreducible automorphic representations $\pi^{(i)}$. Denote by E the surjection from τ_G to θ_G. $E(f^0)$ has a nonzero projection on each summand $\pi^{(i)}$ (and so $\pi^{(i)}$ is unramified at all places). Fix a place ν_0 and consider a decomposable vector in τ_G which at the place ν_0 is arbitrary ξ_{ν_0} (in the space of θ'_{ν_0}) and f^0_ν and the remaining places. Denote such a vector by $j(\xi_{\nu_0})$. Consider the projection of $E(j(\xi_{\nu_0}))$ on $\pi^{(i)}$. This defines a nontrivial map from θ'_{ν_0} to $\pi^{(i)}$ (since $E(j(f^0_{\nu_0})) = E(f^0)$ has a nontrivial projection on $\pi^{(i)}$). Clearly $E(j(\xi_{\nu_0}))$ lies in the subspace of $\prod_{\nu\neq v_0} K_\nu$-fixed vectors of $\pi^{(i)} \simeq \otimes\pi^{(i)}_\nu$, and this is isomorphic, as a G_{ν_0}-module, to $\pi^{(i)}_{\nu_0}$. Thus $\pi^{(i)}_{\nu_0}$ is an irreducible quotient of θ'_{ν_0} and hence, by Proposition 4.2, the following Remark 2 and the last discussion on case D_m, $\pi^{(i)}_{\nu_0} \simeq \theta_{G_{\nu_0}}$ whenever ν_0 is finite. Similarly, when we project $E(j(\xi_{\nu_0}))$ onto $\pi^{(i_1)} \bigoplus \pi^{(i_2)}$, we get that, for $\nu_0 < \infty$, $\pi^{(i_1)}_{\nu_0} \oplus \pi^{(i_2)}_{\nu_0}$ (which is isomorphic to $\theta_{G_{\nu_0}} \oplus \theta_{G_{\nu_0}}$) is a quotient of θ'_{ν_0}. This is impossible unless θ_G is irreducible. ∎

Definition: We call θ_G the automorphic theta representation of $G_{\mathbb{A}}$.

Remark: Although we did not prove that $\theta'_G = \{\mathrm{Res}_{s=s(G)} E^*_{P(G)}(g, f, s)\}$ is irreducible, it is clear, as in the last proof, that θ_G is the unique everywhere unramified irreducible summand of θ'_G. It follows that θ_{G_ν} is a quotient of $\mathrm{Ind}^{G_\nu}_{P(G_\nu)} \delta_{P_\nu}^{s(G)+\frac{1}{2}}$ at all places ν.

From (2.3), (2.5), (2.8) and (2.11), it is easy to deduce, as in Theorem 3.1.

THEOREM 4.4: *We have*

$$\theta^U_G\big|_{L^0(G)_{\mathbb{A}}} = \mathbb{1} \oplus \theta_{L^0(G)}, \qquad G \text{ of type } E_6, E_7, E_8,$$
$$\theta^U_G\big|_{L^0(G)_{\mathbb{A}}} = \mathbb{1} \oplus \left(\theta_{L^0(G)}\right)^\omega, \qquad G \text{ of type } D_m.$$

The action of the center of $L(G)_{\mathbb{A}}$ on $\mathbb{1} \oplus \theta_{L^0(G)}$ is read from (2.4), (2.4), (2.7), (2.10).

5. Fourier coefficients of the theta representation

Let F be a number field and G of type D_m, E_6, E_7, E_8 (simply connected or not). In this section, we consider the Fourier expansion of θ_G along $U(F)\backslash U(\mathbb{A})$. Recall that U is abelian except in case E_8, where U is a Heisenberg group (see (1.2)). The characters of $U(F)\backslash U(\mathbb{A})$ have the following form. Let ψ be a nontrivial character of $F\backslash\mathbb{A}$, and let $Y \in \mathrm{Lie}\ (\overline{U})_F$. Put

$$\psi_Y(\exp Z) = \psi(B(Z, Y)), \qquad Z \in \mathrm{Lie}\ (U)_{\mathbb{A}}.$$

B is the Killing form. In case E_8, we have to assume that Y has zero projection on the root space which corresponds to the negative of the highest root. (In this case, a character of $U(\mathbb{A})$ must be trivial on the center of $U(\mathbb{A})$.) Denote by $\theta_G^{\psi_Y}$ the space of functions on $G(\mathbb{A})$

$$f^{\psi_Y}(g) = \int\limits_{U(F)\backslash U(\mathbb{A})} \psi_Y^{-1}(u) f(ug) du, \tag{5.1}$$

as f varies in θ_G.

Assume that $\theta_G^{\psi_Y}$ is nontrivial. Consider the linear functional

$$\ell_Y(f) = f^{\psi_Y}(1),$$

and choose a finite place ν. By restricting ℓ_Y to θ_{G_ν} (as in the proof of Theorem 4.3) ℓ_Y defines a linear functional $\ell_{Y,\nu}$ on the space $V_{\theta_{G_\nu}}$ of θ_{G_ν}, such that

$$\ell_{Y,\nu}\big(\theta_{G_\nu}(u)\xi\big) = \psi_{Y,\nu}(u)\ell_{Y,\nu}(\xi), \quad u \in U(F_\nu). \tag{5.2}$$

Let us recall, at this point, the notion of a degenerate Whittaker model. Recall (from [MW]) that, for a local (nonarchimedean) field k, a degenerate Whittaker model is defined starting with a nilpotent element $Y \in \mathfrak{g}_k$ and a one-parameter subgroup φ: $k^* \to G(k)$, such that

$$\operatorname{Ad}\varphi(s)Y = s^{-2}Y, \quad s \in k^*. \tag{5.3}$$

Decompose

$$\mathfrak{g}(k) = \oplus \mathfrak{g}_i(k),$$

where

$$\mathfrak{g}_i(k) = \{X \in \mathfrak{g}(k) \mid \operatorname{Ad}\varphi(s)(X) = s^i X\}.$$

Let $N^+(k)$ (resp. $N'(k)$) be the unipotent subgroup of $G(k)$, whose Lie algebra is $\bigoplus_{i\geq 1}\mathfrak{g}_i(k)$ (resp. $\bigoplus_{i\geq 2}\mathfrak{g}_i(k)$). Consider $N''(k)$, the subgroup of $N^+(k)$ generated by $N'(k)$ and the stabilizer, in $N^+(k)$, of Y. Fix ψ, a nontrivial character of k. Then

$$\psi_Y(\exp Z) = \psi(B(Z,Y))$$

defines a character of $N''(k)$.

A smooth irreducible representation π of $G(k)$ is said to have a degenerate Whittaker model relative to (Y,φ), if its Jacquet module with respect to $(N''(k),\psi_Y)$ is nontrivial. The main result in [MW] is that the set of maximal elements of the set of nilpotent orbits which occur in the germ expansion of the character of π is identical with the set of maximal elements of the set of nilpotent orbits which contain an element Y, for which there is φ as in (5.3), so that π has a degenerate Whittaker model relative to (Y, φ).

LEMMA 5.1: *Let k be any field. For every $\alpha \in \Delta(G)$, there is a toral one-parameter subgroup ξ_α, such that*

$$\mathrm{Ad}(\xi_\alpha(a))(X_\gamma) = a^{\delta_{\alpha,\gamma} n_G} X_\gamma \tag{5.4}$$

for $\gamma \in \Delta(G)$ and $a \in k^*$. Here

$$n_G = \begin{cases} 1, & G \text{ of type } E_8, \\ 2, & G \text{ of type } E_7, D_m, \\ 3, & G \text{ of type } E_6. \end{cases}$$

Proof: Write

$$\xi_\alpha(a) = \prod_{\alpha' \in \Delta} h_{\alpha'}(a^{r_{\alpha,\alpha'}}), \quad a \in k^*, \quad r_{\alpha,\alpha'} \in \mathbb{Z}.$$

Then

$$\mathrm{Ad}\,\big(\xi_\alpha(a)\big)(X_\gamma) = a^{\sum_{\alpha' \in \Delta} r_{\alpha,\alpha'}(\alpha',\gamma)} X_\gamma.$$

We then want

$$\sum_{\alpha' \in \Delta} r_{\alpha,\alpha'}(\alpha', \gamma) = \delta_{\alpha,\gamma} n_G,$$

for all $\gamma \in \Delta(G)$, with $r_{\alpha,\alpha'}$ integers. For this, we have to examine the inverse to the Cartan matrix of G, and check that the common denominator of its coordinates is n_G. ∎

We denote

$$\varphi_\alpha(a) = \begin{cases} \xi_\alpha(a^2), & G \text{ of type } E_8, \\ \xi_\alpha(a), & G \text{ of type } E_7, E_6, D_m. \end{cases}$$

Then by (5.4), we have

$$\mathrm{Ad}\,\varphi_\alpha(a) X_\gamma = a^{\delta_{\alpha,\gamma} m_G} X_\gamma, \tag{5.5}$$

for $a \in k^*$, $\alpha, \gamma \in \Delta(G)$ and

$$m_G = \begin{cases} 2, & G \text{ of type } E_7, E_8, D_m, \\ 3, & G \text{ of type } E_6. \end{cases}$$

We introduce φ_α in order to conform with [MW] (when G is not of type E_6).

Our first main result in this section says that θ_G has essentially only one nontrivial Fourier coefficient along U.

THEOREM 5.2: *The space* $\theta_G^{\psi_Y}$ *is nontrivial if and only if* $Y = 0$ *or* $Y \in \mathrm{Ad}\big(L(G_F)\big)\big(X_{-\alpha_Q}\big)$. *(See* (1.2) *for definitions.)*

Proof: We prove this theorem by local reasoning, i.e. using the fact that the local components of θ_G are the local theta representations, and, as a matter of fact, we only use one finite place. (Compare with our work [GRS], where we made a similar use of the smallness of the representation at one archimedean place.)

Assume that $\theta_G^{\psi_Y}$ is nontrivial, and fix a finite place ν. The functional ℓ_Y gives rise to the functional $\ell_{Y,\nu}$ on $V_{\theta_{G_\nu}}$, so that (5.2) is satisfied. Let us show that $\ell_{Y,\nu}$ defines a degenerate Whittaker model of θ_{G_ν}. For this, consider the one-parameter subgroup

$$\varphi = \varphi_{\alpha_Q}. \tag{5.6}$$

By (5.5), it follows that

(5.7)
$$\mathfrak{g}_\nu = \begin{cases} \mathfrak{g}_{\nu,-m_G} \oplus \mathfrak{g}_{\nu,0} \oplus \mathfrak{g}_{\nu,m_G}, & G \text{ of type } E_6, E_7, D_m \\ \mathfrak{g}_{\nu,-2m_G} \oplus \mathfrak{g}_{\nu,-m_G} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\nu,m_G} \oplus \mathfrak{g}_{\nu,2m_G}, & G \text{ of type } E_8 \end{cases}$$

and in case $G \neq E_8$

$$\mathfrak{g}_{\nu,-m_G} = \mathrm{Lie}(\overline{U})_{F_\nu}, \quad \mathfrak{g}_{\nu,0} = \mathrm{Lie}\big(L(G)\big)_{F_\nu}, \quad \mathfrak{g}_{\nu,m_G} = \mathrm{Lie}(U)_{F_\nu},$$

while in case $G = E_8$,

$$\begin{aligned} \mathfrak{g}_{\nu,-2m_G} \oplus \mathfrak{g}_{\nu,-m_G} &= \mathrm{Lie}(\overline{U})_{F_\nu}, \\ \mathfrak{g}_{\nu,0} &= \mathrm{Lie}\big(L(G)\big)_{F_\nu}, \\ \mathfrak{g}_{\nu,m_G} \oplus \mathfrak{g}_{\nu,2m_G} &= \mathrm{Lie}(U)_{F_\nu}. \end{aligned} \tag{5.8}$$

Here, we abbreviated $\mathfrak{g}_i(F_\nu)$ to $\mathfrak{g}_{\nu,i}$. Note that for $G = E_8$, $\mathfrak{g}_{\nu,\pm 2m_G} = \mathfrak{g}_{\nu,\pm 4} = F_\nu \cdot X_{\pm\beta}$, where β is the highest root. It follows from (5.7) and (5.8) that for $Y \in \mathrm{Lie}(\overline{U})F_\nu$,

$$N_\nu^+ = N_\nu' = N''_\nu = U(F_\nu). \tag{5.9}$$

Thus, a degenerate Whittaker model with respect to (Y, φ), for $Y \in \mathrm{Lie}(\overline{U})$ and φ as in (5.6), is given by linear functionals which satisfy (5.2). The result of [MW] and the smallness of θ_{G_ν} imply that if $\ell_{Y,\nu}$ is nontrivial, then Y lies in the closure of $\mathrm{Ad}(G_\nu)(X_\beta)$. Since $\mathrm{Ad}(G_\nu)(X_\beta)$ is the minimal (nontrivial) nilpotent orbit in $\mathfrak{g}_\nu$, then $Y = 0$ or $Y \in \mathrm{Ad}(G_\nu)(X_\beta)$. (Recall that the smallness of θ_{G_ν} means that in the germ expansion of θ_{G_ν} only one nontrivial nilpotent orbit occurs, the coadjoint orbit of highest weight.) We remark here that in case E_6, $\varphi(a)$ satisfies (5.3) with a^{-3} instead of a^{-2}. However, since we have (5.8) and (5.9), the definition of a degenerate Whittaker model relative to (Y,φ) can be repeated, and it is easy to check that the result of [MW] follows exactly in the same way for this case as well, and we reach the same conclusion, namely, if $\ell_{Y,\nu}$ in (5.2) is nontrivial, then $Y = 0$ or $Y \in \mathrm{Ad}(G_\nu)(X_\beta)$.

By Proposition 5.3, proved below, it follows that

$$(5.10)\quad \mathrm{Ad}(G_\nu)(X_\beta) \cap \begin{cases} \mathrm{Lie}(\overline{U})_{F_\nu}, & G \neq E_8, \\ \bigoplus\limits_{\alpha=\sum n_i\alpha_i, n_8=1} \mathfrak{g}_{-\alpha}, & G = E_8 \end{cases} = \mathrm{Ad}\left(L(G)_{F_\nu}\right)\left(X_{-\alpha_Q}\right).$$

Thus, if $Y \neq 0$, then $Y \in \mathrm{Ad}\left(L(G)_{F_\nu}\right)\left(X_{-\alpha_Q}\right) = \mathrm{Ad}\left(L^0(G)_{F_\nu}\right)\left(X_{-\alpha_Q}\right)$ $\left(L^0(G) = [L(G), L(G)]\right)$. Let us show that $Y \in \mathrm{Ad}\left(L^0(G)_F\right)\left(X_{-\alpha_Q}\right)$. Let E be the parabolic subgroup of $L^0(G)$, which preserves $\mathfrak{g}_{-\alpha_Q}$. (The Levi part of E is based on $\Delta(G)\backslash\{\alpha_Q, \alpha'_Q\}$, where α'_Q is the simple root adjacent to α_Q in the Dynkin diagram of G.) E acts on $X_{-\alpha_Q}$ by multiplication by a rational character. Let E^1 be the kernel of this character. Thus, the elements of $\mathrm{Ad}\left(L^0(G_\nu)\right)\left(X_{-\alpha_Q}\right)$ are parameterized by $L^0(G_\nu)/E^1_\nu$ and, by the Bruhat decomposition, they are of the form

$$(5.11)\qquad \mathrm{Ad}\left(xwh_{\alpha'_Q}(t^{-1})\right)\left(X_{-\alpha_Q}\right) = t\,\mathrm{Ad}(xw)\left(X_{-\alpha_Q}\right),$$

where w is an element of the Weyl group of $L^0(G)$, and x is of the form $\prod x_\alpha(r_\alpha)$, $r_\alpha \in F_\nu$, and α ranges over the set I_w of positive roots for $L^0(G)$ such that $w^{-1}(\alpha)$ is a root for the opposite radical of E^1. Clearly, w can be taken in $L^0(G)_F$. Denote $\gamma = w(\alpha_Q)$. This is a root in U. Let S_w be the set of simple roots in I_w. We have

$$(5.12)\quad t\,\mathrm{Ad}\Big(\prod_{\alpha\in I_w} x_\alpha(r_\alpha)\Big)\left(X_{-\gamma}\right) = tX_{-\gamma} + t\sum_{\alpha\in S_w} r_\alpha X_{-\gamma+\alpha} + t\sum c_\mu X_{-\gamma+\mu}.$$

In the third term of (5.12), μ runs over certain roots of height larger than one. Clearly, there are no cancellations in (5.12). Since Y is of the form (5.12), we get that t, c_μ and r_α, for $\alpha \in S_w$, are rational (i.e. lie in F). Consider now μ in I_w

of height two. The coefficient c_μ of $X_{-\gamma+\mu}$ is either r_μ or $r_\mu + r_\alpha r_{\alpha'}$; $\alpha, \alpha' \in S_w$, such that $\mu = \alpha + \alpha'$. Since $c_\mu \in F$ and since $r_\alpha \in F$, for all $\alpha \in S_w$, we get that $r_\mu \in F$. We proceed by induction on the height of $\mu \in I_w$. Assume that $r_\mu \in F$ for all μ in I_w of height less than i. Then, for $\mu \in I_w$ of height i, c_μ has the form $r_\mu + \sum r'$, where r' are products of $r_{\mu'}$, with $\mu' \in I_w$ of height less than i. Since $c_\mu \in F$, we conclude, by induction, that $r_\mu \in F$. Thus $Y \in \mathrm{Ad}\left(L^0(G)_F\right)\left(X_{-\alpha_Q}\right)$, as we wanted.

We proved that, for $f \in \theta_G$, $G \neq E_8$,

$$f(g) = f^U(g) + \sum_{\gamma \in E_F^1 \backslash L^0(G)_F} f^{\psi_{X_{-\alpha_Q}}}(\gamma g). \tag{5.13}$$

Note that it is enough to fix ψ, due to the presence in (5.11) of $t \in F^*$. (5.13) is the Fourier expansion of $f(g)$ along U. This expansion depends only on the constant term and on one nontrivial Fourier coefficient, namely that with respect to $\psi_{X_{-\alpha_Q}}$. Thus, if $Y \in \mathrm{Ad}\left(L(G)_F\right)\left(X_{-\alpha_Q}\right)$ and $\theta_G^{\psi_Y} = 0$, then $f(g) = f^U(g)$, for all $g \in G_{\mathbb{A}}$, and all $f \in \theta_G$. This is impossible, since then $f(q) = f(qu)$, for all $q \in Q_{\mathbb{A}}$ and $u \in U_{\mathbb{A}}$. Since $Q_F \backslash Q_{\mathbb{A}}$ is dense in $G_F \backslash G_{\mathbb{A}}$, we get that $f(g) = f(gu)$, for all $g \in G_{\mathbb{A}}$ and $u \in U_{\mathbb{A}}$. This cannot happen (for example, by the Howe–Moore Theorem [HM]). Of course, θ_G^U is nonzero (by Cor. 2.7). Assume $G = E_8$. In this case, (5.13) is replaced by

$$f^Z(g) = f^U(g) + \sum_{\gamma \in E_F^1 \backslash L^0(G)_F} f^{\psi_{X_{-\alpha_Q}}}(\gamma g),$$

where Z is the center of U $\left(\mathrm{Lie}(Z) = \mathfrak{g}_\beta\right)$. If $Y \in \mathrm{Ad}\left(L(G)_F\right)\left(X_{-\alpha_Q}\right)$ and $\theta_G^{\psi_Y} = 0$, then $f^Z(g) = f^U(g)$ for all $g \in G_{\mathbb{A}}$ and $f \in \theta_G$. Let α be the root of height one less than the height of β. α is a root in U. Denote by N_α the root subgroup which corresponds to α. Consider the Fourier expansion of f along the abelian group $N_\alpha Z$. The group $G_{\alpha_8} = \mathrm{SL}(2)$, which corresponds to $\alpha_Q = \alpha_8$, acts by conjugation on the two-dimensional unipotent group $N_\alpha Z$ according to its natural action on F^2 (simply by identifying $x_\alpha(t) x_\beta(s)$ with (t, s)). Accordingly,

$$f(g) = \sum f^{0,\psi}(\gamma g) + f^{N_\alpha Z}(g).$$

Here, γ runs over $N_{\alpha_8} \backslash G_{\alpha_8}$ and

$$\begin{aligned} f^{0,\psi}(g) &= \int_{F^2 \backslash \mathbb{A}^2} f\left(x_\alpha(t) x_\beta(s) g\right) \psi^{-1}(t)\, dt\, ds = \int_{F \backslash \mathbb{A}} f^Z\left(x_\alpha(t) g\right) \psi^{-1}(t)\, dt \\ &= \int_{F \backslash \mathbb{A}} f^U\left(x_\alpha(t) g\right) \psi^{-1}(t)\, dt = \int_{F \backslash \mathbb{A}} f^U(g) \psi^{-1}(t)\, dt = 0. \end{aligned}$$

Thus, $f = f^{N_\alpha Z}$, and hence $f = 0$. This completes the proof of Theorem 5.2. ∎

Consider the end vertex in the Dynkin diagram, which corresponds to α_Q. Let us redenote $\alpha_Q = \alpha_{\sigma(1)}$; $\alpha_{\sigma(2)}$ is the simple root, which corresponds to the vertex adjacent to that of $\alpha_{\sigma(1)}$, $\alpha_{\sigma(3)}$ is the simple root, which corresponds to the vertex adjacent to that of $\alpha_{\sigma(2)}$, and so on. If $\alpha_{\sigma(i_0)}$ corresponds to the first vertex (in this numeration) adjacent to γ_0, the vertex which has three neighbours, then $\alpha_{\sigma(i_0+1)}$ is the simple root which corresponds to the vertex whose only neighbour is the vertex of γ_0. Next, $\alpha_{\sigma(i_0+2)} = \gamma_0$, and for $i > i_0+2$, $\alpha_{\sigma(i)}$ are the simple roots which correspond to the vertices which follow γ_0 in the direction from γ_0 to the end vertex opposite that of α_Q. (For example, for $G = E_8$, we have

$$\sigma(1) = 8,\ \sigma(2) = 7,\ \sigma(3) = 6, \sigma(4) = 5,\ \sigma(5) = 2,\ \sigma(6) = 4,$$
$$\sigma(7) = 3,\ \sigma(8) = 1; \alpha_Q = \alpha_8,\ \gamma_0 = \alpha_4.)$$

For each i, consider the diagram obtained from the Dynkin diagram of G, after removing the vertices which correspond to $\alpha_{\sigma(j)}$, $1 \le j \le i-1$. The new diagram is the Dynkin diagram of a subgroup L_i ($G = L_0$, $L^0(G) = L_1$). Consider in L_{i-1} the maximal parabolic subgroup $Q_i = L_iU_i$, which corresponds to the root $\alpha_{\sigma(i)}$. $U_i = U_i(G)$ is the unipotent radical of $Q_i = Q_i(G)$. Note that $U_i(G)$ is abelian for $i \ge 2$.

Proposition 5.3: *Let k be a field. Then*

(1) $$\mathrm{Ad}(G_k)(X_\beta) \cap \begin{cases} \mathrm{Lie}(\overline{U})_k, & G \neq E_8, \\ \bigoplus_{\alpha=\sum_{i=1}^8 n_i\alpha_i, n_8=1} \mathfrak{g}_{-\alpha}, & G = E_8 \end{cases} = \mathrm{Ad}\left(L(G)_k\right)\left(X_{-\alpha_Q}\right).$$

(2) *For $i > 1$ and $T \in \mathrm{Lie}(\overline{U}_i)_k$,*

$$X_{-\alpha_Q} + T \in \mathrm{Ad}(G_k)(X_\beta) \Longleftrightarrow T = 0.$$

Proof: Denote

$$D_i = \mathrm{Ad}(G_k)(X_\beta) \cap \begin{cases} X_{-\alpha_Q} + \mathrm{Lie}(\overline{U}_i)_k, & i \ge 2, \\ \mathrm{Lie}(\overline{U})_k, & i = 1 \text{ and } G \neq E_8, \\ \bigoplus_{\alpha=\sum_{i=1}^8 n_i\alpha_i, n_8=1} \mathfrak{g}_{-\alpha}(k), & i = 1 \text{ and } G = E_8. \end{cases}$$

We shall omit k from the notation. Consider the decomposition

$$G = \bigcup QwP_{\mathrm{Heis}}, \tag{5.14}$$

where $Q = Q(G)$. Let

$$K = \Delta(G) \backslash \{\alpha_{P_{\mathrm{Heis}}}\}, \quad J = \Delta(G) \backslash \{\alpha_Q\}.$$

Denote the corresponding sets of positive roots by ϕ_K^+ and ϕ_J^+, respectively. Note that in case $G = E_8$, $Q = P_{\mathrm{Heis}}$. By [C, Prop. 2.7.3], the representatives w in (5.14) can be taken to be Weyl elements, such that

$$w(K) \subset \phi^+(G) \quad \text{and} \quad w^{-1}(J) \subset \phi^+(G). \tag{5.15}$$

The set of Weyl elements which satisfy (5.15) is denoted $D_{J,K}$. These are the elements of minimal length in $W_J \backslash W_G / W_K$. (For $S \subset \Delta$, W_S is the Weyl group of the Levi subgroup based on S.) Since $\mathrm{Ad}(P_{\mathrm{Heis}})(X_\beta) = k^* X_\beta$, we have

$$\mathrm{Ad}(G)(X_\beta) = \bigcup_{w \in D_{J,K}} k^* \,\mathrm{Ad}(Q)\big(X_{w(\beta)}\big). \tag{5.16}$$

Let $w \in D_{J,K}$. Assume that $w(\beta) > 0$. If $w(\beta)$ is a root for U, then $X_{w(\beta)} \in U$, and so $\mathrm{Ad}(Q)\big(X_{w(\beta)}\big) \subset \mathrm{Lie}(U)$, and we get no contribution to the intersection D_i. If $w(\beta) \in \phi_J^+$, then $X_{w(\beta)} \in \mathrm{Lie}\big(L(G)\big)$ and hence $\mathrm{Ad}(Q)\big(X_{w(\beta)}\big) \subset \mathrm{Lie}(Q)$. Thus, there is no contribution in the case to D_i. Assume, then, that $w(\beta) < 0$. This implies that $w(\beta)$ is a root which occurs in $\overline{U} = \overline{U}_1$, since otherwise $w(\beta) \in -\phi_J^+$, i.e. $\beta \in -w^{-1}\big(\phi_J^+\big)$. By (5.15) it follows that β is a negative root — a contradiction. So we may write

$$w(\beta) = -n_0 \alpha_Q - \sum_{\alpha \in J} n_\alpha \cdot \alpha \tag{5.17}$$

where $n_0 \geq 1$, $n_\alpha \geq 0$ are integers. Note that when $G \neq E_8$, we have $n_{\alpha_Q} = 1$, for every root $\gamma = \sum_{\alpha \in \Delta} n_\alpha \alpha$ in the radical U, and when $G = E_8$, $n_{\alpha_Q} = 1$, unless $\gamma = \beta$, in which case $n_{\alpha_Q} = 2$. Let us show that only $n_0 = 1$ in (5.17) contributes to D_i. Indeed, the only other case is when $n_0 = 2$, $G = E_8$ and $w(\beta) = -\beta$, and hence

$$\begin{aligned} \mathrm{Ad}(Q)\big(X_{w(\beta)}\big) &= \mathrm{Ad}(Q)\big(X_{-\beta}\big) = \mathrm{Ad}\left(P_{\mathrm{Heis}}\right)\big(X_{-\beta}\big) = \\ &= k^* \,\mathrm{Ad}(U)\big(X_{-\beta}\big) \subset k^* X_{-\beta} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\beta \oplus \Big(\bigoplus \mathfrak{g}_{-\gamma}\Big), \end{aligned}$$

where γ in the last summand runs over certain positive roots, different from β. Thus, the elements of $\mathrm{Ad}(Q)\big(X_{w(\beta)}\big)$ have a nonzero projection on $X_{-\beta}$, and again do not contribute to D_i. Assume that $n_0 = 1$ in (5.17), and we get

$$w^{-1}(\alpha_Q) = -\beta - \sum_{\alpha \in J} n_\alpha w^{-1}(\alpha). \tag{5.18}$$

By (5.15), $w^{-1}(\alpha) \in \phi^+(G)$, for $\alpha \in J$. Since $n_\alpha \geq 0$, we see from (5.18) that $n_\alpha = 0$ for $\alpha \in J$ (otherwise, $w^{-1}(\alpha_Q)$ is a root which is strictly smaller than $-\beta$). Thus

$$w(\beta) = -\alpha_Q.$$

In this case

$$\begin{aligned} k^* \operatorname{Ad}(Q)\big(X_{w(\beta)}\big) &= k^* \operatorname{Ad}(Q)\big(X_{-\alpha_Q}\big) = \operatorname{Ad}(Q)\big(X_{-\alpha_Q}\big) \\ &= \operatorname{Ad}(LU)\big(X_{-\alpha_Q}\big) \subset \operatorname{Ad}(L)\big(X_{-\alpha_Q} + \operatorname{Lie}(L) + \operatorname{Lie}(U)\big) \\ &= \operatorname{Ad}(L)\big(X_{-\alpha_Q}\big) + \operatorname{Lie}(L) + \operatorname{Lie}(U). \end{aligned} \tag{5.19}$$

Thus, in case $i = 1$,

$$D_1 = \operatorname{Ad}(L)\big(X_{-\alpha_Q}\big).$$

Let $i \geq 2$ and $T \in \operatorname{Lie}(\overline{U}_i)$ such that $X_{-\alpha_Q} + T \in D_i$. From (5.19), it follows that

$$X_{-\alpha_Q} + T = \operatorname{Ad}(\gamma)\big(X_{-\alpha_Q} + Z\big)$$

where $\gamma \in L$ and

$$X_{-\alpha_Q} + Z \in \operatorname{Ad}(U)\big(X_{-\alpha_Q}\big). \tag{5.20}$$

We get that

$$\operatorname{Ad}(\gamma)\big(X_{-\alpha_Q}\big) = X_{-\alpha_Q} \tag{5.21}$$

and

$$\operatorname{Ad}(\gamma)(Z) = T. \tag{5.22}$$

The condition (5.21) means that

$$\gamma \in Q_2^1 = \{g \in Q_2 \mid \operatorname{Ad}(g)X_{-\alpha_{\sigma(1)}} = X_{-\alpha_{\sigma(1)}}\}. \tag{5.23}$$

By (5.22), Z is nilpotent and, by (5.20), $Z \in \operatorname{Lie}(U_2)$. By (5.23), $T = \operatorname{Ad}(\gamma)(Z) \in \operatorname{Lie}(U_2)$. Since $T \in \operatorname{Lie}(\overline{U}_i)$, we get $T = 0$. The proof of Proposition 5.3 is complete. ∎

In the next theorem, we prove an invariance property of the Fourier coefficients $\theta_G^{\psi_Y}$. This is another aspect of the smallness and rigidity of θ_G. By Theorem 5.2, it is enough to consider $\theta_G^{\psi_{X_{-\alpha_Q}}}$. Denote

$$R = E^1 U.$$

(E^1 is defined in the proof of Theorem 5.2. Note that $E = Q_2$.) R is almost the full stabilizer of $\psi_{X_{-\alpha_Q}}$ in Q (a one-dimensional torus is "missing"). Thus, we have

$$f^{\psi_{X_{-\alpha_Q}}}(rg) = f^{\psi_{X_{-\alpha_Q}}}(g), \quad \forall r \in R(F)$$

for any automorphic form on $Q(\mathbb{A})$. The automorphic form of θ_G satisfies the following very strong property.

THEOREM 5.4: *For all $f \in \theta_G$,*

$$f^{\psi_{X_{-\alpha_Q}}}(rg) = f^{\psi_{X_{-\alpha_Q}}}(g), \quad \forall r \in R(\mathbb{A}). \tag{5.24}$$

Proof: We show that the Fourier expansion of $f^{\psi_{X_{-\alpha_Q}}}$ along U_i, $i \geq 2$, contains only the constant term, and we do it step by step. Consider the Fourier expansion of $f^{\psi_{X_{-\alpha_Q}}}$ along U_2. The characters of $U_2(F)\backslash U_2(\mathbb{A})$, which appear in the expansion, have the form

$$\psi_T(\exp V) = \psi(B(V,T)), \qquad V \in \mathrm{Lie}(U_2)_{\mathbb{A}},$$

where $T \in \mathrm{Lie}(\overline{U}_2)_F$. The corresponding Fourier coefficient of $f^{\psi_{X_{-\alpha_Q}}}$ is a Fourier coefficient of f along the group U_1U_2 with respect to the character

$$\psi_{X_{-\alpha_Q}+T}(\exp V) = \psi\big(B(V, X_{-\alpha_Q} + T)\big), \qquad V \in \mathrm{Lie}(U_1)_{\mathbb{A}} \oplus \mathrm{Lie}(U_2)_{\mathbb{A}}.$$

Denote this Fourier coefficient by $f^{\psi_{X_{-\alpha_Q}+T}}$. We will use [MW] as in Theorem 5.2 to show that $f^{\psi_{X_{-\alpha_Q}+T}} = 0$, unless $T = 0$. Indeed, fix a finite place ν and regard $f^{\psi_{X_{-\alpha_Q}+T}}(I)$ as a linear functional on θ_{G_ν}. Denote this functional by $\ell^{(2)}_{T,\nu}$. We have

$$\ell^{(2)}_{T,\nu}\big(\theta_{G_\nu}(u)\xi\big) = \psi_{X_{-\alpha_Q}+T}(u)\ell^{(2)}_{T,\nu}(\xi) \tag{5.25}$$

for $u \in (U_1U_2)_{\mathbb{A}}$, $\xi \in V_{\theta_{G_\nu}}$. $\ell^{(2)}_{T,\nu}$ defines a degenerate Whittaker model for θ_{G_ν}. This model is relative to $\big(X_{-\alpha_Q} + T, \varphi\big)$, where φ is the one-parameter subgroup

$$\varphi(a) = \varphi_{\alpha_{\sigma(2)}}(a)\varphi_{\alpha_{\sigma(1)}}(a). \tag{5.26}$$

By (5.5), we have for a root $\gamma = \sum_{\alpha\in\Delta} n_\alpha \cdot \alpha$,

$$\mathrm{Ad}\,\varphi(a)X_\gamma = a^{m_G\left(n_{\alpha_{\sigma(1)}}+n_{\alpha_{\sigma(2)}}\right)}X_\gamma. \tag{5.27}$$

Recall our notation

$$\mathfrak{g}_{\nu,i} = \{X \in \mathfrak{g}_\nu \mid \mathrm{Ad}\,\varphi(a)X = a^i X\}.$$

Then by (5.27) $X_{-\alpha_Q} + T \in \mathfrak{g}_{\nu,-m_G}$, and moreover,

$$\bigoplus_{i \geq m_G} \mathfrak{g}_{\nu,i} = \mathrm{Lie}(U_1)_{F_\nu} \oplus \mathrm{Lie}(U_2)_{F_\nu}, \tag{5.28}$$

$$\bigoplus_{i \leq -m_G} \mathfrak{g}_{\nu,i} = \mathrm{Lie}\big(\overline{U}_1\big)_{F_\nu} \oplus \mathrm{Lie}\big(\overline{U}_2\big)_{F_\nu}; \tag{5.29}$$

$\mathfrak{g}_\nu$ is the sum of the spaces $\mathfrak{g}_{\nu,0}$ and those of (5.28) and (5.29). Thus, as in (5.9), we have

$$N_\nu^+ = N_\nu' = N_\nu'' = \big(U_1U_2\big)_{F_\nu}.$$

(The case E_6, is treated exactly as in Theorem 5.2, i.e. the main result of [MW] applies in this case exactly in the same way.) We conclude that

$$X_{-\alpha_Q} + T \in \mathrm{Ad}(G_\nu)\big(X_\beta\big)$$

if $\ell_{T,\nu}^{(2)}$ is nonzero. By Proposition 5.3, it follows that $T = 0$. Assume, by induction, that for all $2 \leq j \leq i-1$ and all $T \in \mathrm{Lie}\big(\overline{U}_j\big)_F$,

$$f^{\psi_{X_{-\alpha_Q}+T;j}}(g) = \int\limits_{U_j(F)\backslash U_j(\mathbb{A})} f^{\psi_{X_{-\alpha_Q}}}(ug)\psi_T^{-1}(u)du$$

is identically zero, unless $T = 0$. ($\psi_T(\exp V) = \psi(B(V,T))$ for $V \in \mathrm{Lie}(U_j)_{\mathbb{A}}$.) We prove that $f^{\psi_{X_{-\alpha_Q}+T;i}} = 0$ for all $T \in \mathrm{Lie}\big(\overline{U}_i\big)_F$, unless $T = 0$. Let $T \in \mathrm{Lie}\big(U_i\big)_F$. Fix a finite place ν, and, again, consider $f^{\psi_{X_{-\alpha_Q}+T;i}}(I)$ as a linear functional $\ell_{T,\nu}^{(i)}$ on θ_{G_ν}. It satisfies

$$\ell_{T,\nu}^{(i)}\big(\theta_{G_\nu}(u)\xi\big) = \psi_{X_{-\alpha_Q}+T}^{(i)}(u)\ell_{T_\nu}^{(i)}(\xi)$$

for $u \in \big(U_1U_2 \cdot \ldots \cdot U_i\big)_{F_\nu}$ and $\xi \in V_{\theta_{G_\nu}}$. $\psi_{X_{\alpha_Q}+T}^{(i)}$ is the character of $\big(U_1U_2 \cdot \ldots \cdot U_i\big)_{F_\nu}$

$$\psi_{X_{-\alpha+Q}+T}^{(i)}(\exp V_1 \cdot \ldots \cdot \exp V_i) = \psi\big(B(V_1, X_{-\alpha_Q})\big)\psi(B(V_i,T)),$$

for $V_j \in \mathrm{Lie}(U_j)_{F_\nu}$; $1 \leq j \leq i$. $\ell_{T,\nu}^{(i)}$ defines a degenerate Whittaker model for Q_{G_ν}, relative to $\big(X_{-\alpha_Q} + T, \varphi\big)$, where φ is the one-parameter subgroup

$$\varphi(a) = \varphi_{\alpha_{\sigma(1)}}(a) \cdot \varphi_{\alpha_{\sigma(2)}}(a) \cdot \ldots \cdot \varphi_{\alpha_{\sigma(i)}}(a).$$

By (5.5), we have for a root $\gamma = \sum_{\alpha\in\Delta} n_\alpha \cdot \alpha$,

$$\mathrm{Ad}\,\varphi(a)X_\gamma = a^{m_G\left(n_{\alpha_{\sigma(1)}}+n_{\alpha_{\sigma(2)}}+\cdots+n_{\alpha_{\sigma(i)}}\right)}X_\gamma.$$

Thus, $X_{-\alpha_Q} + T \in \mathfrak{g}_{\nu,-m_G}$ and also

$$\bigoplus_{j\geq m_G} \mathfrak{g}_{\nu,j} = \bigoplus_{j=1}^{i} \mathrm{Lie}(U_j)_{F_\nu}, \tag{5.30}$$

$$\bigoplus_{j\leq -m_G} \mathfrak{g}_{\nu,j} = \bigoplus_{j=1}^{i} \mathrm{Lie}(\overline{U}_j)_{F_\nu}; \tag{5.31}$$

$\mathfrak{g}_\nu$ is the sum of $\mathfrak{g}_{\nu,0}$ and the spaces in (5.30) and (5.31). As in (5.9), we conclude that

$$N_\nu^+ = N_\nu' = N_\nu'' = \left(U_1U_2\cdot\ldots\cdot U_i\right)_{F_\nu}$$

and that

$$X_{-\alpha_Q} + T \in \mathrm{Ad}\left(G_\nu\right)\left(X_\beta\right).$$

By Proposition 5.3, this implies that $T = 0$. We have shown that $f^{\psi_{X_{-\alpha_Q}}}$ is left invariant under the adele points of the standard maximal unipotent subgroup of G. Now the theorem follows, using (5.23). ∎

6. On the theta representation of SO_{2m}

In this section, we prove the uniqueness (multiplicity one) of the automorphic theta representation of SO_{2m}. We will abbreviate and write θ_m instead of $\theta_{\mathrm{SO}_{2m}}$. As a result, we will exhibit θ_m as a family of residues of degenerate Eisenstein series induced on the parabolic subgroup of SO_{2m}, which preserves an isotropic line.

THEOREM 6.1: *Let π be an irreducible automorphic representation of $\mathrm{SO}_{2m}(\mathbb{A})$ (which acts on the space V_π). Assume that π is isomorphic to θ_m. Then $\pi = \theta_m$ (i.e. $V_\pi = V_{\theta_m}$).*

Proof: Since π is isomorphic to θ_m, the proof of Theorem 5.2 applies to π as well. (All we needed there was that there is a (finite) place ν such that $\pi_\nu \simeq \theta_{m,\nu}$.) Thus, in the Fourier expansion of π along $U_m = U(\mathrm{SO}_{2m})$, the only characters which appear there are ψ_Y, where $Y = 0$ or $Y \in \mathrm{Ad}\left(L(\mathrm{SO}_{2m})_F\right)\left(X_{-\alpha_{Q(\mathrm{SO}_{2m})}}\right)$. Let us rewrite this in matrix notation,

$$Y \in \mathrm{Ad}\begin{pmatrix} 1 & & \\ & \gamma & \\ & & 1 \end{pmatrix}(X_{-\beta_m}), \quad \gamma \in \mathrm{SO}_{2m-2}(F),$$

and

$$\psi_Y\begin{pmatrix} 1 & x & * \\ & I_{2m-2} & x' \\ & & 1 \end{pmatrix} = \psi_\gamma\begin{pmatrix} 1 & x & * \\ & I_{2m-2} & x' \\ & & 1 \end{pmatrix} = \psi\left((x\cdot\gamma)_1\right).$$

Here, for $y \in \mathbb{A}^{2m-2}$, $(y)_i$ (or sometimes y_i) denotes the i-th coordinate of y. As in (5.13), the Fourier expansion of $f \in \pi$ is

$$f(g) = f^{U_m}(g) + \sum_{\gamma \in Q^0_{m-1}(F)\backslash \mathrm{SO}_{2m-2}(F)} f^\psi(\widehat{\gamma}g). \tag{6.1}$$

Here $\widehat{\gamma} = \begin{pmatrix} 1 & & \\ & \gamma & \\ & & 1 \end{pmatrix}$, Q^0_{m-1} is the stabilizer in SO_{2m-2} of β_m (Q^0_{m-1} is the stabilizer in SO_{2m-2} of $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ under the natural left action of SO_{2m-2} on the $2(m-1)$ dimensional space). Note that $Q_{m-1} = Q(\mathrm{SO}_{2m-2}) = h_{\beta_{m-1}}(F^*)Q^1_{m-1}$. ψ in (6.1) is short for $\psi_{X_{-\beta_m}}$. By Theorem 5.4, f^ψ satisfies the following invariance property

$$f^\psi(rg) = f^\psi(g) \tag{6.2}$$

for all $r \in R_{\mathbb{A}}$, where $R = \widehat{Q}^0_{m-1}U_m$ $\left(\widehat{Q}^0_{m-1} = \left\{\begin{pmatrix} 1 & & \\ & \gamma & \\ & & 1 \end{pmatrix} \mid \gamma \in Q^0_{m-1}\right\}\right)$. Let $T\colon \pi \longrightarrow \theta_m$ be an isomorphism. We will show below (in Theorem 6.2) that for a place ν the space of linear functionals on $\theta_{m,\nu}$, such that

$$\ell\big(\theta_{m,\nu}(r)\xi\big) = \psi_\nu(r)\ell(\xi), \tag{6.3}$$

is one dimensional. In (6.3), $r \in R_\nu$ and $\psi_\nu(r)$ is defined as $\psi_{X_{-\beta_m},\nu}$ on $U_{m,\nu}$ and is extended trivially to $\widehat{Q}^0_{m-1,\nu}$. If ν is archimedean, ℓ is assumed to be continuous in the C^∞-topology. As a result, we conclude that there is a nonzero complex number c, such that

$$(T(f))^{\psi}(g) = cf^{\psi}(g) \tag{6.4}$$

for all $f \in \pi$ and $g \in \mathrm{SO}_{2m}(\mathbb{A})$. By (6.1) and (6.4),

$$T(f) - cf = (T(f) - cf)^{U_m}. \tag{6.5}$$

If $T \neq c\cdot\mathrm{id}$, then $V' = \{T(f) - cf \mid f \in \pi\}$ defines an automorphic representation of $\mathrm{SO}_{2m}(\mathbb{A})$, which is isomorphic to θ_m. The elements of V' satisfy, by (6.5), $\xi = \xi^{U_m}$. As in the end of the proof of Theorem 5.2, it follows (using the Howe–Moore theorem) that $\xi \equiv 0$. Thus $T = c\cdot\mathrm{id}$ and so $\pi = \theta_m$. ∎

The main ingredient of the proof of Theorem 6.1 is then the uniqueness, up to scalar multiples, of the functionals (6.3) at every place ν.

THEOREM 6.2: *Let ν be a place of F. Then the space of linear functionals on $\theta_{m,\nu}$, which satisfy (6.3), and continuous in the C^∞-topology, in case ν is archimedean, is one dimensional.*

Proof: Since $\theta_{m,\nu}$ is a quotient of $I_{m,\nu} = \mathrm{Ind}_{P_{m,\nu}}^{\mathrm{SO}_{2m}(F_\nu)} \delta^{s_m+\frac{1}{2}}$, $s_m = \frac{m-3}{2m-2}$ ($P_m = P(\mathrm{SO}_{2m})$), it is enough to show the same uniqueness statement on $I_{m,\nu}$. This will be shown using Bruhat theory. We will do the archimedean case only. The finite case is similar and is much simpler. We omit reference to ν in the course of this proof. Put $I_m(s) = \mathrm{Ind}_{P_m}^{\mathrm{SO}_{2m}} \delta_{P_m}^{s+\frac{1}{2}}$ and consider

$$\mathrm{Hom}_R(I_m(s), \psi) \simeq \mathrm{Bil}_R\left(I_m(s), \psi^{-1}\right) \simeq \mathrm{Bil}_{\mathrm{SO}_{2m}}\left(I_m(s), \mathrm{Ind}_R^{c\,\mathrm{SO}_{2m}} \psi^{-1}\right).$$

Bil_G denotes G-equivariant continuous bilinear forms, Ind_R^c denotes compact $(\mathrm{mod}\, R)$ induction. The last isomorphism is by Frobenius reciprocity. Note that R is unimodular. By Bruhat theory [W, Theorem 5.3.2.3],

$$\dim\left(\mathrm{Bil}_{D_m}\left(I_n(s), \mathrm{Ind}_R^{c\,D_m} \psi^{-1}\right)\right) \leq$$
$$\sum_{\gamma \in P_m \backslash D_m / R} \sum_{k=0}^{\infty} \dim\left(\mathrm{Bil}_{\gamma^{-1}P_m\gamma \cap R}\left((\delta_{P_m}^{1/2})^\gamma \delta_\gamma^{-1} \Lambda_{\gamma,k}^\vee, \psi^{-1} \otimes (\delta_{P_m}^s)^\gamma\right)\right).$$

Denote each summand by $i_{\gamma,k}(s)$. Here $(\delta_{P_m}^s)^\gamma(b) = \delta_{P_m}^s(\gamma b \gamma^{-1})$, for $b \in \gamma^{-1}P_m\gamma \cap R$; δ_γ is the modular factor for the group $\gamma^{-1}P_m\gamma \cap R$; $\Lambda_{\gamma,k}^\vee$ is the dual of $\Lambda_{\gamma,k} = \mathrm{Sym}^k(\Lambda_{\gamma,1})$, where $\Lambda_{\gamma,1}$ is the coadjoint action of $\gamma^{-1}P_m\gamma \cap R$ on

$$\mathcal{B}_\gamma = \frac{\mathrm{Lie}(\mathrm{SO}_{2m})_{\mathbb{C}}}{\mathrm{Lie}(R)_{\mathbb{C}} + \mathrm{Ad}(\gamma^{-1})\mathrm{Lie}(P_m)_{\mathbb{C}}};$$

$\Lambda_{\gamma,0}=1$. $P\backslash \mathrm{SO}_{2m}/R$ consists of four elements. We pick the following representatives: I, w_0', w_0, w_0w_0', where

$$w_0'=w_{\beta_{m-1}}w_{\beta_{m-2}}\cdot\ldots\cdot w_{\beta_1}w_{\beta_3}w_{\beta_4}\cdot\ldots\cdot w_{\beta_{n-1}}=\begin{pmatrix} 1 & & & & & & \\ & 0 & \cdots & \cdots & \cdots & 1 & \\ & \vdots & I_{m-3} & & & \vdots & \\ & \vdots & & 0\ \ 1 & & \vdots & \\ & \vdots & & 1\ \ 0 & & \vdots & \\ & \vdots & & & I_{m-3} & \vdots & \\ & 1 & \cdots & \cdots & \cdots & 0 & \\ & & & & & & 1 \end{pmatrix},$$

$$w_0=w_{\beta_m}w_{\beta_{m-1}}\cdot\ldots\cdot w_{\beta_1}w_{\beta_3}w_{\beta_4}\cdot\ldots\cdot w_{\beta_m}=\begin{pmatrix} 0 & \cdots & \cdots & \cdots & 1 \\ \vdots & I_{m-2} & & & \vdots \\ \vdots & & 0\ \ 1 & & \vdots \\ \vdots & & 1\ \ 0 & & \vdots \\ \vdots & & & I_{m-3} & \vdots \\ 1 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

For $\gamma=1$ or $\gamma=w_0'$, $\gamma^{-1}P_m\gamma\cap R\supset U_m$. We have

$$\psi^{-1}\otimes\left(\delta_{P_m}^{s}\right)^{\gamma}\big|_{U_m}=\psi\big|_{U_n}\quad\text{and}\quad\left(\delta_{P_m}^{1/2}\right)^{\gamma}\delta_{\gamma}^{-1}\big|_{U_m}=1.$$

Now since $\Lambda_{\gamma,k}$ is algebraic, $\Lambda_{\gamma,k}\big|_{U_m}$ is unipotent and so has no nontrivial eigenvalues (on U_m). We conclude that $i_{\gamma,k}(s)=0$ for all k and s. Similarly, for $\gamma=w_0w_0'$, we have

$$\gamma^{-1}P_m\gamma\cap R\supset V=\left\{\begin{pmatrix} 1 & v & & & \\ & 1 & & & \\ & & I_{2m-2} & & \\ & & & 1 & -v \\ & & & & 1 \end{pmatrix}\right\}.$$

Since $\psi\big|_V$ is nontrivial, we get, exactly as in the previous two cases, that $i_{\gamma,k}(s)=0$ for all k and s. Let $\gamma=w_0$. We have

$$w_0^{-1}P_mw_0\cap R=\left\{a(h,y,*)=\left(\begin{array}{c|c|cc|c|c} 1 & 0 & \cdots 0\ y_1 & \cdots & y_{m-1} & 0 \\ \hline & 1 & * \cdots * & \cdots & * & y'_{m-1} \\ \hline & & & & & \vdots \\ & & h & * & * & y'_1 \\ & & & & \vdots & 0 \\ & & 0 & h^* & * & \vdots \\ \hline & & & & 1 & 0 \\ \hline & & & & & 1 \end{array}\right)^{\omega}\right\}.$$

h in $a(h, y, *)$ is in GL_{m-2} and $y = (y_1, \ldots, y_{m-1})$; ω is the matrix

$$\begin{pmatrix} I_{m-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{m-1} \end{pmatrix}$$

and for a matrix b, $b^\omega = \omega^{-1} b \omega$. We have

$$\begin{aligned} \left(\delta_{P_m}^s\right)^{w_0}\left(a(h,y,*)\right) &= |\det h|^{(m-1)s}, \\ \delta_{w_0}^{-1}\left(a(h,y,*)\right) &= |\det h|^{2-m}, \\ \psi\big|_{w_0^{-1}P_m w_0 \cap R} &= 1. \end{aligned}$$

Thus, we consider, using a loose notation,

$$\mathrm{Bil}_{w_0^{-1}P_m w_0\cap R}\left(|\det h|^{(m-1)s}\ ,\ |\det h|^{\frac{3-m}{2}}\Lambda^\vee_{w_0,k}\right). \tag{6.6}$$

The quotient $\mathcal{B}_{w_0}$ is isomorphic to

$$\left\{ b(v) = \left(\begin{array}{ccccc|c} & & \underline{0} & & & \underline{0} \\ \hline 0 & v_1 & & & & \\ \vdots & \vdots & & \underline{0} & & \\ 0 & v_{m-2} & & & & \underline{0} \\ 0 & 0 & -v_{m-2} & \cdots & -v_1 & \\ 0 & 0 & 0 & \cdots & 0 & \end{array}\right) \middle|\, v = \begin{pmatrix} v_1 \\ \vdots \\ v_{m-2} \end{pmatrix} \in F^{m-2} \right\}$$

so that the action of $a(h,0,0)$ through $\Lambda_{w_{0,1}}$ on $b(v)$ is via $v \mapsto h^* v$, and hence, via $\Lambda_{w_{0,k}}$, it is through $\mathrm{Sym}^k(h^*)$. In particular, the space (6.6) is zero, for all s and all $k \geq 1$, i.e. $i_{w_0,k}(s) = 0$, for all s and all $k \geq 1$. For $k = 0$, the space (6.6) is nonzero, if and only if $(m-1)s = -\frac{3-m}{2}$, i.e. $s = s_m$. Thus, $\mathrm{Hom}_R(I_m(s), \psi) = 0$ for all $s \neq s_m$, and for $s = s_m$ it is of dimension at most one. This dimension equals one, since θ_m is a quotient of $I_m(s_m)$, and we know that $\mathrm{Hom}_R(\theta_m, \psi)$ has positive dimension (using the global Fourier coefficient f^ψ in (6.1)). This concludes the proof of Theorem 6.2. ∎

Remark 1: One can show directly that $\mathrm{Hom}_R(I_m(s_m), \psi)$ is one dimensional, by constructing the following linear functional. Define, for ξ_s, a holomorphic section in $I_m(s)$,

$$A(\xi_s)(g) = \int\limits_{U'_m} \xi_s(w_0 ug)\psi^{-1}(u)du,$$

$$U'_m = \left\{ u(y) = \begin{pmatrix} 1 & y & 0 & 0, \\ & I_{m-1} & 0 & 0 \\ & & I_{m-1} & y' \\ & & & 1 \end{pmatrix}^{\omega} \middle| y \in F^{m-1} \right\},$$

$$\psi(u(y)) = \psi(y_1), \qquad \text{for } y = (y_1, \dots, y_{m-1}).$$

We have

$$A(\xi_s)(ug) = \psi(u)A(\xi_s)(g)$$

for u in the unipotent radical of R, and

$$A(\xi_s)\left(\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \begin{pmatrix} h & x \\ 0 & h^* \end{pmatrix} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}^{\omega} g\right) = |\det h|^{(m-1)s + \frac{m-3}{2}} A(\xi_s)(g).$$

In particular, for $s = s_m$, $|\det h|^{(m-1)s_m + \frac{m-3}{2}} = |\det h|^{m-3} = \delta_{P_{m-2}} \begin{pmatrix} h & x \\ 0 & h^* \end{pmatrix}$, i.e. $A(\xi_{s_m}) \begin{pmatrix} I_2 & & \\ & g & \\ & & I_2 \end{pmatrix} \in I_{m-2}(\frac{1}{2})$, and this representation has the trivial representation (of D_{m-2}) as a quotient. Composing A with this quotient yields an element of $\mathrm{Hom}_R(I_m(s_m), \psi)$. All this is of course formal, but it can be justified by writing

$$A(\xi_s)(g) = \int M_{w_{\beta_3} w_{\beta_4} \cdot \dots \cdot w_{\beta_{m-1}}} \circ M_{w_{\beta_2}}(\xi_s)\big(w_{\beta_m} x_{\beta_m}(y) g\big)\psi^{-1}(y)dy; \tag{6.7}$$

M_w denotes an intertwining integral. The poles of $M_{w_{\beta_2}}$ are contained in those of $\zeta_{F_\nu}(2(m-1)s+m-2)$, and so $M_{w_{\beta_2}}$ is holomorphic at $s = s_m$; $M_{w_{\beta_3} \cdot \dots \cdot w_{\beta_{m-1}}}$ is holomorphic at s_m as well. This can be seen using Rallis' Lemma as in Lemma 2.6. Finally, the integral (6.7) is a GL_2-Whittaker type integral and it is known to be holomorphic. ∎

Remark 2: Theorem 6.2 holds, with the same proof, for a group of type D_m. The arguments prior to Theorem 6.2 are general, and so Theorem 6.1 holds for a group of type D_m as well. ∎

We return to the global situation. Consider the induced representation $J_m(s) = \mathrm{Ind}_{Q_m(\mathbb{A})}^{D_m(\mathbb{A})} \delta_{Q_m}^{s+\frac{1}{2}}$. Denote $\widetilde{h}(t_1, \ldots, t_m) = \mathrm{diag}(t_1, \ldots, t_m, t_m^{-1}, \ldots, t_1^{-1})$. Then $\delta_{Q_m}\left(\widetilde{h}(t_1, \ldots, t_m)\right) = |t_1|^{2m-2}$. Let $f(g,s)$ be a holomorphic section in $J_m(s)$ and

$$E_{Q_m}(g, f, x) = \sum_{\gamma \in Q_m(F)\backslash \mathrm{SO}_{2m}(F)} f(\gamma g, s)$$

the corresponding Eisenstein series. As explained in Section 2, the normalizing factor is $L_S(D_m, Q_m, s) = \zeta_S((2m-2)s+1)\zeta_S((2m-2)s+m-1)$ where S is a finite set of places, containing those at infinity, and outside which f is unramified. Let $E^*_{Q_m}(g,f,s) = L_S(D_m, Q_m, s)E_{Q_m}(g,f,s)$, the normalized Eisenstein series. As in Theorem 2.3, we have

Proposition 6.3: *For $n \geq 4$, $E^*_{Q_m}(g,f,s)$ has at most a simple pole at the point $s'_m = \frac{1}{2m-2}$, and it is obtained for some choice of section f.*

Proof: As in the proof of Theorem 2.3, since $Q_m\backslash \mathrm{SO}_{2m}/Q_m$ has three representatives $\{I, w_{\beta_m}, w_0\}$, we have, for $g = \begin{pmatrix} a & & \\ & h & \\ & & a^{-1} \end{pmatrix} \in \mathrm{SO}_{2m}(\mathbb{A})$, $a \in \mathbb{A}^*$,

$$\begin{aligned} E_{Q_m}^{U_m}(g,f,s) = & \int_{U_m(F)\backslash U_m(\mathbb{A})} E_{Q_m}(ug,f,s)du \\ = & |a|^{(2m-2)(s+1/2)} f(I,s) \\ & + |a| E_{Q_{m-1}}\left(M_{w_{\beta_m}}(s)f, h, \frac{m-1}{m-2}s\right) \\ & + |a|^{(2m-2)(-s+\frac{1}{2})} M_{w_0}(s)f(I). \end{aligned} \tag{6.8}$$

In the second term of (6.8), we restrict $M_{w_{\beta_m}}(s)f$ to $\mathrm{SO}_{2m-2}(\mathbb{A})$; $M_{w_{\beta_m}}(s)f|_{D_{m-1}(\mathbb{A})} \in J_{m-1}(\frac{m-1}{m-2}s)$. We have

$$\begin{aligned} & L_S(D_m, Q_m, s)E_{Q_{m-1}}\left(M_{w_{\beta_m}}(s)f, h, \frac{m-1}{m-2}\right) \\ = & L_S\left(D_{m-1}, Q_{m-1}, \frac{m-1}{m-2}s\right) \\ & \times E_{Q_{m-1}}\left(\frac{\zeta_S((2m-2)s+m-1)}{\zeta_S((2m-2)s+m-2)} M_{w_{\beta_m}}(s)f, h, \frac{m-1}{m-2}s\right) \\ = & E^*_{Q_{m-1}}\left(A_{w_{\beta_m}}(s)f, h, \frac{m-1}{m-2}s\right). \end{aligned} \tag{6.9}$$

Here, we use the notations of Section 2. We also have

(6.10)
$$\begin{aligned} L_S(D_m, Q_m, s)M_{w_0}(s)f &= \zeta_S((2m-2)s)\zeta_S((2m-2)s-m+2))A_{w_0}(s)f \\ &= \zeta((2m-2)s)\zeta((2m-2)s-m+1)A^*_{w_0}(s)f. \end{aligned}$$

Here $A^*_{w_0}(s)f = \prod_{\nu\in S}(\zeta_\nu((2m-2)s)\zeta_\nu((2m-2)s-m+2))^{-1}A_{w_0}(s)f$. Multiplying (6.8) by the normalizing factor, and using (6.9), (6.10), we get, for
$$g = \begin{pmatrix} a & & \\ & h & \\ & & a^{-1}\end{pmatrix},$$

$$\begin{aligned} \left(E^*_{Q_m}\right)^{U_m}(f,g,s) =&|a|^{(2m-2)(s+\frac{1}{2})}L_S(D_m,Q_m,s)f(I,s) \\ &+|a|E^*_{Q_{m-1}}\left(A_{w_{\beta_m}}(s)f, h, \frac{m-1}{m-2}s\right) \\ &+|a|^{(2m-2)(-s+\frac{1}{2})}\zeta((2m-2)s) \\ &\times\zeta((2m-2)s-m+2)A^*_{w_0}(s)f(I). \end{aligned} \tag{6.11}$$

We will soon show that $A_{w_{\beta_m}}(s)$ and $A^*_{w_0}(s)$ are holomorphic at $s=s'_m$. Consider the case $m=4$. Using the triviality of D_4, we can write

$$E_{Q_4}(1,f,s) = E_{P(\mathrm{SO}_8)}(1,f^\tau,s)$$

where $f^\tau \in \mathrm{Ind}_{P_4(\mathbb{A})}^{\mathrm{SO}_8(\mathbb{A})}\delta_{P_4}^{s+1/2}$; $f^\tau(g) = f(g^\tau)$ and τ is an outer automorphism (coming from triviality), which takes β_4 to β_2, β_2 to β_1, β_1 to β_4 and fixes β_3. Note that $s'_4=\frac{1}{6}=s(D_4)$. By Theorem 2.3 for this case (or rather [KR1]) $E^*_{Q_4}$ has a simple pole at s'_4. By induction, it follows from (6.11) that $E^*_{Q_m}$ has at most a simple pole at s'_m, for $m\geq 4$. (Note that $\frac{m-1}{m-2}s'_m = s'_{m-1}$.) Substitute $f=f_0^{(m)}$, the everywhere normalized unramified vector. Then (6.11) reads

$$\begin{aligned} \left(E^*_{Q_n}\right)^{U_m}(g,f_0,s) =&|a|^{(2m-2)(s+\frac{1}{2})}L_\phi(D_m,Q_m,s) \\ &+|a|E^*_{Q_{m-1}}\left(f_0^{(m-1)}, h, \frac{m-1}{m-2}s\right) \\ &+|a|^{(2m-2)(-s+\frac{1}{2})}s((2m-2)s) \\ &\cdot\zeta((2m-2)s-m+2). \end{aligned} \tag{6.12}$$

The first term in (6.12) is holomorphic, the third term has a simple pole at $s=s'_m$, and the second term has also a simple pole at s'_m (by induction). The residues of the second term and the third term do not cancel due to their different exponents in a. Finally, as in the proofs of Lemmas 2.4–2.6, it is enough to prove the holomorphicity of $M_{w,\nu}(s)f_\nu$ at s'_m at a place ν ($w = w_{\beta_m}, w_0$). For $w = w_{\beta_m}$, the poles of $M_{w,\nu}(s)f_\nu$ are contained in those of $\zeta_\nu\big(\langle -\rho + (s+\frac{1}{2})2\rho_{Q_m}, \beta_m\rangle\big) = \zeta_\nu((2m-2)s+m-2)$, which is holomorphic at s'_m. As for $w = w_0$, recall that

$$A^*_{w_0,\nu}(s)f_\nu = (\zeta_\nu((2m-2)s)\zeta_\nu((2m-2)s-m+1))^{-1}M_{w_0,\nu}(s)f_\nu.$$

The factor $\zeta_\nu((2m-2)s)$ may be ignored, since it contributes $\zeta_\nu(1)$ at $s = s'_m$. Write

$$M_{w_0,\nu}(s) = M_{w_{\beta_2}w_{\beta_3}\cdot\ldots\cdot w_{\beta_m}},\nu \circ M_{w_{\beta_1},\nu} \circ M_{w_{\beta_3},\nu} \circ M_{w_{\beta_4},\nu} \circ \ldots \cdot M_{w_{\beta_m},\nu}.$$

The poles of $\prod_{i\neq ?} M_{w_{\beta_i},\nu}$ are contained in those of $\prod_{j=0}^{m-2}\zeta_\nu((2m-2)s+j)$, which is holomorphic at s'_m. Put $w' = w_{\beta_2}w_{\beta_3}\cdot\ldots\cdot w_{\beta_m}$. Then

$$w' = \left(\begin{array}{cc|cc} 0 & I_{m-1} & & \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & & I_{m-1} & 0 \end{array}\right)^{\omega} \qquad \left(\omega = \begin{pmatrix} I_{m-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{m-1} \end{pmatrix}\right).$$

Thus $w' \in P(\mathrm{SO}_{2m})^\omega \equiv P^\omega_m = M(\mathrm{SO}_{2m})^\omega \cdot V(D_m)^\omega \equiv M^\omega_m \cdot V^\omega_m$. Put $\varphi = M_{w_{\beta_1},\nu} \cdot M_{w_{\beta_3},\nu} \cdot \ldots \cdot M_{w_{\beta_m},\nu}(f)$ and consider $\varphi\big|_{M^\omega_m}$. Let $\widetilde{\varphi}_s = \big(\varphi\big|_{M^\omega_m}\big)^\omega$ and identify M_m with GL_m. Then $\widetilde{\varphi}_s \in \mathrm{Ind}^{\mathrm{GL}_m(F_\nu)}_{P_{m-1,1}}\left(\begin{pmatrix} a & * \\ 0 & t\end{pmatrix} \to |\det a|\ |t|^{-(2m-2)s}\right)$. $P_{m-1,1}$ is the $(m-1,1)$ type parabolic subgroup of GL_m. Thus, we have to consider the poles of

$$\int \widetilde{\varphi}_s\left(\begin{pmatrix} 0 & I_{m-1} \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & I_{m-1}\end{pmatrix}\right)dz = \int \varphi'_s\begin{pmatrix} I_{m-1} & 0 \\ z & 1\end{pmatrix}dz. \tag{6.13}$$

Here $\widetilde{\varphi}_s(g) = \varphi'_s\left(\begin{pmatrix} 0 & 1 \\ I_{m-1} & 0\end{pmatrix}\right)$. By Rallis' Lemma (as in Lemma 2.6), we may assume that φ'_s is supported in the open cell $P_{m-1,1}\begin{pmatrix} 0 & I_{m-1} \\ 1 & 0\end{pmatrix}P_{m-1,1}$. For such functions, it is easy to see that the integral (6.12) can be written in the form

$$\int_{F^*_\nu}\int_{F^{m-2}_\nu} \phi(v,x)|x|^{(2m-2)s-m+2}dv\,d^*x = \int_{F^*_\nu} \Phi(x)|x|^{(2m-2)s-m+2}d^*x \tag{6.14}$$

where $\phi \in S\big(F^{m-2}_\nu \times F_\nu\big)$ and $\Phi(x) = \int_{F^{m-2}_\nu}\phi(v,x)dv$. Thus, the integral (6.14) is controlled (as far as poles are concerned) by $\zeta_\nu((2m-2)s-m+2)$ as we wanted. Proposition 6.3 is now proved. ∎

Denote by $\widetilde{\theta}_m$ the space of residues at s'_m of $E^*_{Q_m}$. From (6.11), we get

COROLLARY 6.4: *We have*

$$(\widetilde{\theta}_m)^{U_m}|_{D_{m-1}(\mathbb{A})} \subset 1 \oplus \widetilde{\theta}_{m-1} \tag{6.15}$$

and $h(a) = \widetilde{h}(a, 1, \ldots, 1)$ *acts on* 1 *by* $|a|^{m-2}$ *and on* $\widetilde{\theta}_{m-1}$ *by* $|a|$.

Since $\widetilde{\theta}_m$ is concentrated along the Borel subgroup, it follows from Corollary 6.4 that $\widetilde{\theta}_m$ has the same exponents as those of θ_m and, by Proposition 3.1, we conclude

COROLLARY 6.5: $\widetilde{\theta}_m$ *consists of square integrable automorphic forms.*

Remark: The analogs of Proposition 6.3, Corollary 6.4 and Corollary 6.5 are clear for groups of type D_m. See Remark 2.7.

Write $\widetilde{\theta}_m = \bigoplus_i \widetilde{\theta}_m^{(i)}$ as a direct sum of irreducible, automorphic representations $\widetilde{\theta}_m^{(i)} \simeq \bigotimes_\nu \widetilde{\theta}_{m,\nu}^{(i)}$. Clearly $\widetilde{\theta}_{m,\nu}^{(i)}$ is a quotient of $J_{m,\nu}(s_m')$ for each i (and each ν). By the results of [S], for finite ν, and of [HT], for ν archimedean, it follows that $J_{m,\nu}(s_m')$ has a unique quotient and it is unramified. Thus

COROLLARY 6.6: $\widetilde{\theta}_m$ *is irreducible.*

PROPOSITION 6.7: *For all places* ν,

$$\theta_{m,\nu} \simeq \widetilde{\theta}_{m,\nu}.$$

Proof: We construct an intertwining operator from $I_{m,\nu}(s_m))$ to $J_{m,\nu}(s_m')$. Let f_s be a holomorphic section in $I_m(s)$. Then, for $g \in \mathrm{SO}_{2m-2,\nu}$, $\varphi(g) = f_{s_m}\begin{pmatrix} 1 & & \\ & g & \\ & & 1 \end{pmatrix}$ lies in $I_{m-1}(\frac{1}{2})$, which has the trivial representation of $\mathrm{SO}_{2m-2,\nu}$ as a quotient. Let T be a $\mathrm{SO}_{2m-2,\nu}$-invariant linear form on $I_{m-1}(\frac{1}{2})$ and consider $A'(f_{s_m}) = T(\varphi)$. Since $f_s\begin{pmatrix} t & * & * \\ & I & * \\ & & t^{-1} \end{pmatrix} = |t|^{(m-1)(s+\frac{1}{2})} f_s(I_{2m})$, it follows that A' induces (by considering $g \mapsto A(g \cdot f_{s_m})$ on $\mathrm{SO}_{2m,\nu}$) an intertwining map $A\colon I_m(s_m) \longrightarrow J_m(-s_m')$. Clearly, A is nontrivial on the unramified vector of $I_m(s_m)$, which is cyclic for $I_m(sm)$. Thus the image of A is unramified. By the results of [S] for ν finite and of [HT], for ν archimedean, $J_m(-s_m')$ has a unique irreducible subrepresentation, and it is unramified. This representation is $\widetilde{\theta}_{m,\nu}$. Thus $\widetilde{\theta}_{m,\nu}$ is a quotient of $I_{m,\nu}(s_m)$ and hence $\widetilde{\theta}_{m,\nu} \simeq \theta_{m,\nu}$. ∎

From Theorem 6.1, we conclude

THEOREM 6.8:

$$\theta_m = \{\mathrm{Res}_{s=s'_m} E_{Q_m}(f,g,s) | f(\cdot,s) \textit{ holomorphic section in } J_m(s)\}.$$

Remark: Consider the analogous automorphic representation $\widetilde{\theta}_m^{sc}$ of Spin($\mathbb{A}$), obtained by the residues of $E^*_{Q_m^{sc}}$ at s'_m (analogous Eisenstein series on Spin($\mathbb{A}$)). As in Remark 2.7, $E_{Q_m^{sc}}(g, i^*(\varphi_s)) = E_{Q_m}(i(g), \varphi_s)$, for a secton φ_s in $g_m(s)$. By Theorem 6.8 and the discussion prior to Theorem 4.3, $\widetilde{\theta}_m$ is irreducible over $i(\mathrm{Spin}(\mathbb{A}))$. This implies that Span$\{g \to \mathrm{Res}_{s=s'_m} E^*_{Q_m}(i(g), \varphi_s)\}$ affords an irreducible representation of Spin($\mathbb{A}$). Hence $\widetilde{\theta}_m^{sc}$ is irreducible and equals θ_m^{sc}. As we have seen before, we conclude that for a group G of type D_m, if we construct the analogous representation $\widetilde{\theta}_G$ by the residues of $E^*_{Q(G)}$ at s'_m, then $\widetilde{\theta}_G$ equals θ_G as automorphic representations.

Finally, let us relate $\mathrm{Res}_{s=s'_m} E_{Q_m}(f,g,s)$ to the θ-lift of the trivial representation of $\mathrm{SL}_2(\mathbb{A})$ to $\mathrm{SO}_{2m}(\mathbb{A})$ (via the dual pair $\mathrm{SO}_{2m} \times \mathrm{SL}_2$ inside Sp_m). This is explained in [KR2], where (in this special case) they consider the Weil representation $\omega_\psi^{(m)}$ of $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$ (rank $2m$) and restrict it to the dual pair $\mathrm{SO}_{2m}(\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A})$ (ψ is a nontrivial character of $F \backslash \mathbb{A}$). $\omega_\psi^{(m)}$ acts on $S(\mathbb{A}^{2m})$, and $\mathrm{SO}_{2m}(\mathbb{A})$ acts on $S(\mathbb{A}^{2m})$, through $\omega_\psi^{(m)}$, by the natural linear action. Let $\phi \in S(\mathbb{A})^{2m})$ and consider the theta series

$$\theta_\psi^{(m)}(g,h;\phi) = \sum_{x \in F^{2m}} \omega_\psi^{(m)}(g,h)\phi(x), \quad g \in \mathrm{SO}_{2m}(\mathbb{A}), \quad h \in \mathrm{SL}_2(\mathbb{A}).$$

Let

$$E(h,s) = \sum_{\gamma \in B_F \backslash \mathrm{SL}_2(F)} |a(\gamma h)|^{s+1}$$

where, for $h \in \mathrm{SL}_2(\mathbb{A})$,

$$h = \begin{pmatrix} a(h) & * \\ 0 & a(h)^{-1} \end{pmatrix} k(h)$$

is the Iwasawa decomposition. B is the Borel subgroup of SL_2. $E(h,s)$ is the unramified Eisenstein series on $\mathrm{SL}_2(\mathbb{A})$ (given by the convergent series for $\mathrm{Re}(s) > 1$ and by its meromorphic continuation of $\mathrm{Re}(s) \leq 1$). Consider

$$(6.16) \quad I(g,s,\omega_\psi^{(m)}(1,\Omega'_m)\phi) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \theta_\psi^{(m)}(g,h;\omega_\psi^{(m)}(1,\Omega'_m)\phi)E(h,s)dh$$

where $\Omega'_m = \Omega_m - m^2 + 2m$ and $\Omega_m = H^2 - 2H + 4X_+X_-$ is the Casimir element for SL_2 at one fixed archimedean place (with the usual notation). It is proved in [KR2, Prop. 5.3.1] that $h \mapsto \theta_\psi^{(m)}(g, h; \omega_\psi(1, \Omega'_m)\phi)$ is rapidly decreasing on $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})$. Moreover, the function (6.16) (divided by $s^2 - (m - 1^2)$) is equal to an Eisenstein series of the form $E_{Q_m}\big(f, g, \frac{s}{2m-2}\big)$ (f depends on ϕ) [KR2, 5.5.23]. Taking residue at $s = 1$, we get, using Theorem 6.8,

THEOREM 6.9:

$$\theta_m = \Big\{ \int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})} \theta_\psi^{(m)}(g, h; \omega_\psi^{(m)}(1, \Omega'_m)\phi)dh \mid \phi \in S(\mathbb{A}^{2m}) \Big\}.$$

Remark: Let i: Spin $\longrightarrow \mathrm{SO}_{2m}$ be the central isogeny. From Theorem 6.9, it follows that

$$\theta_m^{sc} = \Big\{ g \mapsto \int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})} \theta_\psi^{(m)}\big(i(g), h; \omega_\psi^{(m)}(1, \Omega'_m)\phi\big)dh \mid \phi \in S(\mathbb{A}^{2m}) \Big\}. \tag{6.17}$$

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